

Isoperimetric domains of large volume in homogeneous three-manifolds

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Abstract

Given a non-compact, simply connected homogeneous three-manifold X and a sequence $\{\Omega_n\}_n$ of isoperimetric domains in X with volumes tending to infinity, we prove that as $n \rightarrow \infty$:

1. The radii of the Ω_n tend to infinity.
2. The ratios $\frac{\text{Area}(\partial\Omega_n)}{\text{Vol}(\Omega_n)}$ converge to the Cheeger constant $\text{Ch}(X)$, which we also prove to be equal to $2H(X)$ where $H(X)$ is the critical mean curvature of X .
3. The values of the constant mean curvatures H_n of the boundary surfaces $\partial\Omega_n$ converge to $\frac{1}{2}\text{Ch}(X)$.

Furthermore, when $\text{Ch}(X)$ is positive, we prove that for n large, $\partial\Omega_n$ is well-approximated in a natural sense by the leaves of a certain foliation of X , where every leaf of the foliation is a surface of constant mean curvature $H(X)$.

Keywords. Surface with constant mean curvature, critical mean curvature, isoperimetric domain, foliation, metric Lie group, homogeneous three-manifold, Cheeger constant

1 Introduction.

Throughout this paper, X **will denote a non-compact, simply connected homogeneous three-manifold**. Recall that the *isoperimetric profile* of X is defined as the function

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$I: (0, \infty) \rightarrow (0, \infty)$ given by

$$I(t) = \inf \{ \text{Area}(\partial \mathcal{D}) : \overline{\mathcal{D}} \subset X \text{ is a smooth compact domain with } \text{Volume}(\mathcal{D}) = t \}. \quad (1.1)$$

The function $I(t)$ has been extensively studied; see the background Section 2 for a brief summary of some properties of $I(t)$. For every value of $t \in (0, \infty)$, there exists at least one smooth compact domain $\Omega \subset X$ of volume t and area $I(t)$, and this domain has boundary of non-negative constant mean curvature with respect to the inward pointing unit normal vector, due to the fact that X is homogeneous. Such a smooth compact domain Ω with smallest boundary area for its given volume is called an *isoperimetric domain* of X .

In this paper we will study the geometry of isoperimetric domains in X of large volume. The following definitions provide three key geometric invariants which we will study in detail in order to describe the geometry of these isoperimetric domains.

Definition 1.1. Let \mathcal{A} be the collection of all compact, immersed orientable surfaces in X , and given a surface $\Sigma \in \mathcal{A}$, let $|H_\Sigma|: \Sigma \rightarrow [0, \infty)$ stand for the absolute mean curvature function of Σ . The *critical mean curvature* of X is defined as

$$H(X) = \inf \{ \max_{\Sigma} |H_\Sigma| : \Sigma \in \mathcal{A} \}. \quad (1.2)$$

Definition 1.2. The *radius* of a compact Riemannian manifold with boundary is the maximum distance from points in the manifold to its boundary.

One goal of this paper is to prove Theorem 1.4 below which relates the mean curvatures of boundaries of isoperimetric domains with large volume to the Cheeger constant of X , defined as follows.

Definition 1.3. The *Cheeger constant* of a Riemannian manifold Y with infinite volume is

$$\text{Ch}(Y) = \inf \left\{ \frac{\text{Area}(\partial \mathcal{D})}{\text{Vol}(\mathcal{D})} : \overline{\mathcal{D}} \subset Y \text{ is a smooth compact domain} \right\}. \quad (1.3)$$

By definition of the Cheeger constant, $\text{Ch}(X) = \inf \{ \frac{I(t)}{t} \mid t \in (0, \infty) \}$ for every non-compact, simply connected homogeneous three-manifold X , where I is the isoperimetric profile of X .

Theorem 1.4. *Let X be a non-compact, simply connected homogeneous three-manifold.*

(1) *Suppose that X is not isometric to the Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ of a two-sphere of constant curvature $\kappa > 0$ with the real line. If $\Omega \subset X$ is an isoperimetric domain in X with volume t , then $\partial\Omega$ is connected and*

$$\text{Ch}(X) < \min \left\{ 2H, \frac{I(t)}{t} \right\},$$

where $H > 0$ is the constant mean curvature of the boundary of Ω with respect to the inward pointing unit normal.

$$(2) \text{Ch}(X) = 2H(X) = \lim_{t \rightarrow \infty} \frac{I(t)}{t}.$$

- (3) *Given any sequence of isoperimetric domains $\Omega_n \subset X$ with volumes tending to infinity, as $n \rightarrow \infty$, the sequence of constant mean curvatures of their boundaries converges to $H(X)$ and the sequence of radii of these domains diverges to infinity.*

Theorem 1.4 provides additional information about the isoperimetric profile of a non-compact, simply connected homogeneous three-manifold, see Corollary 6.4.

The organization of the paper is as follows. In Section 2 we provide some background information about the isoperimetric profile of a non-compact, simply connected, homogeneous three-manifold X , and prove some parts of items (1) and (2) of Theorem 1.4. The rest of the proof of Theorem 1.4 is divided into two cases; in Section 3 we consider the case when the Cheeger constant of X is zero, and in Sections 4, 5, 6 we will deal with the case when $\text{Ch}(X) > 0$. When $\text{Ch}(X)$ is positive, we prove in Theorems 6.2 and 6.3 that the boundaries of isoperimetric domains of large volume in X are in a natural sense well-approximated by the leaves of a certain foliation \mathcal{F} of X , where these leaves are surfaces of constant mean curvature $H(X)$. As a matter of fact, we will prove in Section 6 the following existence result.

Theorem 1.5. *Let X be a homogeneous three-manifold diffeomorphic to \mathbb{R}^3 . Then, there exists a foliation \mathcal{F} of X by surfaces of constant mean curvature $H(X)$ with the following properties.*

- (1) *There exist a 1-parameter subgroup Γ and a $(\mathbb{Z} \times \mathbb{Z})$ -subgroup Δ of the isometry group $\text{Iso}(X)$ of X , both acting freely on X , such that each of the leaves of \mathcal{F} is invariant under Γ and Δ .*
- (2) *All leaves of \mathcal{F} are congruent in X ; more precisely, there exists a 1-parameter subgroup $\tilde{\Gamma}$ of $\text{Iso}(X)$ acting freely on X such that $\mathcal{F} = \{\phi(\Sigma) \mid \phi \in \tilde{\Gamma}\}$, where Σ is any particular leaf of \mathcal{F} .*
- (3) *Each orbit of the action of $\tilde{\Gamma}$ on X intersects every leaf of \mathcal{F} transversely at a single point.*

Our interest in results like the ones described above arises from our paper [4], where we classify the moduli space of constant mean curvature spheres in a homogeneous three-manifold X in terms of $H(X)$. The work in [4] and in the present paper support the following conjecture:

Conjecture 1.6 (Uniqueness of Isoperimetric Domains Conjecture). *If X is a homogeneous three-manifold diffeomorphic to \mathbb{R}^3 , then for each $V > 0$, there exists a unique (up to congruencies) isoperimetric domain $\Omega(V)$ in X with volume V . Furthermore, $\Omega(V)$ is topologically a compact ball.*

Several results in this paper admit generalizations to the n -dimensional case, where one assumes that $n \leq 7$ in order to avoid lack of regularity of the boundaries of isoperimetric domains. Nevertheless, for the sake of simplicity we will develop here only the case where the ambient homogeneous manifold is three-dimensional.

2 Background material.

2.1 Isoperimetric profile.

Since X is homogeneous and the dimension of X is less than 8, then by regularity and existence results in geometric measure theory, for each $t > 0$, there exists at least one smooth compact domain $\Omega \subset X$ that is a solution to the isoperimetric problem in X with volume t , i.e., whose boundary surface $\partial\Omega$ minimizes area among all boundaries of domains in X with volume t ; as usual, we call such Ω an *isoperimetric domain*. It is well-known that the (possibly non-connected) boundary $\partial\Omega$ of an isoperimetric domain Ω has constant mean curvature. In the sequel, we will always orient $\partial\Omega$ with respect to the inward pointing unit normal vector of Ω .

For any $\varepsilon > 0$ and $t \geq \varepsilon$, there exist uniform estimates for the norm of the second fundamental forms of the isoperimetric surfaces³ enclosing volume t , see e.g., the proof of Theorem 18 in the survey paper by Ros [10] for a sketch of proof of this curvature estimate.

Consider the isoperimetric profile $I: (0, \infty) \rightarrow (0, \infty)$ of X defined in (1.1). This profile has been extensively studied in greater generality. We next recall some basic properties of it, see e.g., Bavard and Pansu [1], Gallot [2] and Ros [10]:

- (I) I is locally Lipschitz. In particular, its derivative I' exists almost everywhere in $(0, \infty)$ and for every $0 < t_0 \leq t_1$,

$$I(t_1) - I(t_0) = \int_{t_0}^{t_1} I'(t) dt.$$

- (II) I has left and right derivatives $I'_-(t)$ and $I'_+(t)$ for any value of $t \in (0, \infty)$. Moreover if H is the mean curvature of an isoperimetric surface $\partial\Omega$ with $\text{Volume}(\Omega) = t$ (with the notation above), then $I'_+(t) \leq 2H \leq I'_-(t)$.

- (III) The limit as $t \rightarrow 0^+$ of $\frac{I(t)}{(36\pi t^2)^{1/3}}$ is 1.

Remark 2.1. (i) Every isoperimetric domain Ω in a non-compact homogeneous three-manifold is connected; otherwise translate one component of Ω along a continuous

³An *isoperimetric surface* is the boundary of an isoperimetric domain.

1-parameter family of isometries until it touches another component tangentially a first time to obtain a contradiction to boundary regularity of solutions to isoperimetric domains.

- (ii) From the discussion at the beginning of Section 2.2, if X is not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, then X is diffeomorphic to \mathbb{R}^3 . Suppose that X is diffeomorphic to \mathbb{R}^3 and Ω is an isoperimetric domain in X . As Ω is a connected compact domain in $X \approx \mathbb{R}^3$, the boundary $\partial\Omega$ contains a unique outer boundary component ∂ (here, outer means the component of the boundary of Ω that is contained in the boundary of the unbounded component of $X - \Omega$). We claim that $\partial\Omega = \partial$ and that $\partial\Omega$ has positive mean curvature with respect to the inward pointing normal vector to Ω . Since connected, compact embedded surfaces in \mathbb{R}^3 bound unique smooth compact regions, we can translate a superimposed copy of the compact region $\Omega(\partial) \subset X$ enclosed by ∂ along a 1-parameter group of ambient isometries until the translated copy intersects $\Omega(\partial)$ a last time at some point p . At this last point p of contact, the outer boundaries of the two intersecting domains intersect on opposite sides of their common tangent plane at p . The maximum principle for constant mean curvature surfaces applied at p demonstrates that the mean curvature of ∂ is positive with respect to the inward pointing normal of Ω . Another simple continuous translation argument of a possible interior boundary component ∂' of Ω , applied in the interior of $\Omega(\partial)$, gives a contradiction that implies that the boundary of an isoperimetric domain Ω is equal to its outer boundary component; hence $\partial\Omega$ is connected.
- (iii) In the case that X is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, similar arguments to those appearing in item (ii) show that the boundary of an isoperimetric domain Ω in X is either connected with constant positive mean curvature, or Ω is the product domain $\mathbb{S}^2(\kappa) \times [R_1, R_2]$ for some $R_1 < R_2$. In fact, Pedrosa [8] proved that there exists $V_0 > 0$ such that if $\Omega \subset X$ is an isoperimetric domain with $\text{Vol}(\Omega) = V > 0$, then Ω is a rotationally symmetric ball if $V < V_0$, and $\Omega = \mathbb{S}^2(\kappa) \times [R_1, R_2]$ if $V > V_0$ for some $R_1 < R_2$. In particular, the isoperimetric profile of X is constant in the interval $[V_0, \infty)$.

The main goal of this section is to prove the next four lemmas.

Lemma 2.2. *Given $t \in (0, \infty)$, $\frac{1}{2}I'_+(t)$ (resp. $\frac{1}{2}I'_-(t)$) equals the infimum (resp. supremum) of the mean curvatures of isoperimetric surfaces in X enclosing volume t . In fact, this infimum (resp. supremum) is achieved for some isoperimetric domain. Furthermore, the function I is non-decreasing and strictly increasing when X is diffeomorphic to \mathbb{R}^3 .*

Proof. We will prove the stated properties for the case of $I'_+(t)$ and leave the similar case of $I'_-(t)$ to the reader. Fix $t_0 \in (0, \infty)$ and let \mathcal{B} be the family of isoperimetric

surfaces in X enclosing volume t_0 . Since I is locally Lipschitz, then I is differentiable in $[t_0, t_0 + 1] - A$, where $A \subset [t_0, t_0 + 1]$ is a set of measure zero, and the function $t \in [t_0, t_0 + 1] \mapsto I'(t)$ is integrable in $[t_0, t_0 + 1]$.

We claim that there exists a sequence $t_n \in (t_0, t_0 + 1] - A$ converging to t_0 such that $I'(t_n) < I'_+(t_0) + \frac{1}{n}$; otherwise, there exist numbers $\varepsilon, \delta > 0$ such that $I'(t) \geq I'_+(t_0) + \delta$ for all $t \in (t_0, t_0 + \varepsilon) - A$. This inequality implies that for $t \in (t_0, t_0 + \varepsilon)$,

$$\frac{I(t) - I(t_0)}{t - t_0} = \frac{1}{t - t_0} \int_{t_0}^t I'(t) dt \geq I'_+(t_0) + \delta,$$

which contradicts the existence of the right derivative of I at t_0 and therefore, proves our claim.

Given $n \in \mathbb{N}$, there exists an isoperimetric domain $\Omega_n \subset X$ with volume t_n . As $t_n \in (t_0, t_0 + 1] - A$, then I is differentiable at t_n and property (II) stated just before Remark 2.1 implies that $I'(t_n) = 2H_n$, where H_n denotes the constant mean curvature of $\partial\Omega_n$. Since X is homogeneous, standard compactness results imply that after possibly passing to a subsequence, the Ω_n converge to an isoperimetric domain $\Omega_\infty \subset X$ enclosing volume t_0 , and the sequence of mean curvatures H_n of their boundaries converge to the mean curvature H_∞ of $\partial\Omega_\infty \in \mathcal{B}$ as $n \rightarrow \infty$. Thus, by the claim proved above, $I'_+(t_0) + \frac{1}{n} > 2H_n$ and taking $n \rightarrow \infty$, $I'_+(t_0) \geq 2H_\infty \geq 2 \inf\{H(\Sigma) \mid \Sigma \in \mathcal{B}\}$, where $H(\Sigma)$ denotes the constant mean curvature of $\Sigma \in \mathcal{B}$. The inequality $I'_+(t_0) \leq 2 \inf\{H(\Sigma) \mid \Sigma \in \mathcal{B}\}$ follows directly from property (II) above, which completes the proof of the first sentence of the lemma. The second statement (that the infimum is achieved) holds since $I'_+(t_0) = 2H_\infty$, which is the mean curvature of the isoperimetric domain Ω_∞ .

Recall from Remark 2.1 that the mean curvature of the boundary of every isoperimetric domain in X is non-negative, and it is positive when X is diffeomorphic to \mathbb{R}^3 . As for every $t > 0$, $I'_+(t)$ is twice the mean curvature of some isoperimetric domain in X enclosing volume t , then $0 \leq I'_+(t) \leq I'_-(t)$ for every $t > 0$, from where one deduces that I is non-decreasing. In the case X is diffeomorphic to \mathbb{R}^3 , the same argument gives that I is strictly increasing and the proof is complete. \square

Lemma 2.3. *With the notation above,*

- (1) *Given $t > 0$ and $n \in \mathbb{N}$, we have $I(nt) < nI(t)$.*
- (2) *$\text{Ch}(X) < \frac{I(t)}{t}$ for all $t > 0$.*

Proof. Given $t > 0$, let $\Omega \subset X$ be an isoperimetric domain with volume t . Since Ω is compact and X is non-compact and homogeneous, there exists an isometry ϕ of X such that $\phi(\Omega)$ is disjoint from Ω . As $\Omega \cup \phi(\Omega)$ has volume $2t$ but it is not an isoperimetric domain (since it is not connected), then

$$I(2t) < \text{Area}[\partial(\Omega \cup \phi(\Omega))] = 2 \text{Area}(\partial\Omega) = 2I(t).$$

Item (1) follows by applying this argument to a collection $\{\Omega, \phi_1(\Omega), \phi_2(\Omega), \dots, \phi_{n-1}(\Omega)\}$ of pairwise disjoint translated copies of Ω .

By definition of $\text{Ch}(X)$, we have $\text{Ch}(X) \leq \frac{I(t)}{t}$ for each t . If $\text{Ch}(X)$ were equal to $\frac{I(t_0)}{t_0}$ for some $t_0 > 0$, then item (1) of this lemma would imply that

$$\frac{I(2t_0)}{2t_0} < \frac{2I(t_0)}{2t_0} = \frac{I(t_0)}{t_0} = \text{Ch}(X),$$

which is absurd. This proves $\text{Ch}(X) < \frac{I(t)}{t}$ for all $t > 0$. \square

Lemma 2.4. *For any $\varepsilon > 0$, there exists $V_\varepsilon > 0$ such that for every isoperimetric domain $\Omega \subset X$ with volume greater than V_ε ,*

$$\text{Ch}(X) < \frac{\text{Area}(\partial\Omega)}{\text{Vol}(\Omega)} < \text{Ch}(X) + \varepsilon. \quad (2.1)$$

Proof. The first inequality in (2.1) holds for every isoperimetric domain by item (2) of Lemma 2.3.

Given $\varepsilon > 0$, consider a compact domain $\mathcal{D}_0 \subset X$ with volume V_0 , such that $\text{Ch}(X) < \frac{\text{Area}(\partial\mathcal{D}_0)}{V_0} < \text{Ch}(X) + \frac{\varepsilon}{2}$. Such domain \mathcal{D}_0 exists by definition of $\text{Ch}(X)$ and by item (2) of Lemma 2.3. Observe that $I(V_0) \leq \text{Area}(\partial\mathcal{D}_0) < V_0(\text{Ch}(X) + \frac{\varepsilon}{2})$. Consider the piecewise constant function $F: (0, \infty) \rightarrow \mathbb{R}$ given by

$$F(t) = (k+1)V_0(\text{Ch}(X) + \frac{\varepsilon}{2}) \text{ if } t \in (kV_0, (k+1)V_0], \text{ for any } k \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. By item (1) of Lemma 2.3, $I((k+1)V_0) < (k+1)I(V_0) < F((k+1)V_0)$. By Lemma 2.2 I is non-decreasing, and so, $I \leq F$ in $(kV_0, (k+1)V_0]$ for every k ; thus, $I \leq F$ in $(0, \infty)$.

On the other hand, a straightforward computation shows that the function F lies below the linear function $t > 0 \mapsto (\text{Ch}(X) + \varepsilon)t$ for $t \geq 2 + \frac{2}{\varepsilon}\text{Ch}(X)$. Finally, $\frac{I(t)}{t} \leq \frac{F(t)}{t} \leq \text{Ch}(X) + \varepsilon$ for $t \geq 2 + \frac{2}{\varepsilon}\text{Ch}(X)$ and the proof is complete. \square

Lemma 2.5. *For each $n \in \mathbb{N}$, there exists $T_n > n$ such that for every isoperimetric domain $\Omega_n \subset X$ with volume T_n , the mean curvature H_n of its boundary satisfies $H_n < \frac{1}{2}\text{Ch}(X) + \frac{1}{n}$. In particular, $2H(X) \leq \text{Ch}(X)$.*

Proof. By Lemma 2.4, given $n \in \mathbb{N}$ there exists $t_n > 0$ such that for every isoperimetric domain $\Omega \subset X$ with volume greater than t_n , we have

$$\text{Ch}(X) < \frac{\text{Area}(\partial\Omega)}{\text{Vol}(\Omega)} < \text{Ch}(X) + \frac{1}{n}. \quad (2.2)$$

Clearly we can assume $t_n > n$ without loss of generality.

We claim that for each n there exists $T_n \in (t_n, \infty)$ such that the left derivative $I'_-(t)$ of I satisfies $I'_-(T_n) < \text{Ch}(X) + \frac{2}{n}$. Arguing by contradiction, suppose that $I'_-(t) \geq \text{Ch}(X) + \frac{2}{n}$ for all $t \in (t_n, \infty)$. Then, given $t > t_n$ we have

$$I(t) - I(t_n) = \int_{t_n}^t I'(s) ds \geq \left[\text{Ch}(X) + \frac{2}{n} \right] (t - t_n) \quad (2.3)$$

Using (2.2) and (2.3), we have

$$\text{Ch}(X) + \frac{1}{n} > \frac{I(t)}{t} \geq \left[\text{Ch}(X) + \frac{2}{n} \right] \left(1 - \frac{t_n}{t} \right) + \frac{I(t_n)}{t}. \quad (2.4)$$

Taking $t \rightarrow \infty$ (with n fixed) in (2.4) and simplifying we obtain $\frac{1}{n} \geq \frac{2}{n}$, which is absurd. Hence, our claim holds.

We finish by proving that the statement of the lemma holds for the value T_n found in the last paragraph. Given an isoperimetric domain $\Omega_n \subset X$ with volume T_n , we can apply property (II) stated just before Remark 2.1 to the mean curvature H_n of the boundary $\partial\Omega_n$ and then apply the claim in the last paragraph in order to get

$$2H_n \leq I'_-(T_n) < \text{Ch}(X) + \frac{2}{n},$$

which completes the proof of the lemma. \square

2.2 Classification of the ambient manifolds.

As X is three-dimensional, simply connected and homogeneous, then X is isometric either to the Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ of a 2-sphere of constant curvature $\kappa > 0$ with the real line or to a *metric Lie group*, i.e., a Lie group equipped with a left invariant metric (see e.g., Theorem 2.4 in [5]). A special case of this second possibility is that X is isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some 2×2 real matrix A endowed with its *canonical metric*; this means that the group structure is given on $\mathbb{R}^2 \times \mathbb{R}$ by the operation

$$(\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{z_1 A} \mathbf{p}_2, z_1 + z_2), \quad (2.5)$$

where e^{zA} is the usual exponentiation of matrices, and the canonical left invariant metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is the one that extends the usual inner product in $\mathbb{R}^3 = T_e(\mathbb{R}^2 \rtimes_A \mathbb{R})$ by left translation with respect to the above group operation, where $e = (0, 0, 0)$. Every three-dimensional, simply connected non-unimodular Lie group lies in this semidirect product case, as well as the unimodular groups $\tilde{\text{E}}(2)$ (the universal cover of the group of orientation-preserving rigid motions of the Euclidean plane), Sol_3 (also known as $E(1, 1)$, the group of orientation-preserving rigid motions of the Lorentz-Minkowski plane) and Nil_3 (the Heisenberg group of nilpotent 3×3 real upper triangular matrices with entries 1 in the diagonal).

The classification of three-dimensional, simply connected Lie groups ensures that except for the cases listed in the above paragraph, the remaining Lie groups are $SU(2)$ (the unitary group, diffeomorphic to the three-sphere) and $\widetilde{SL}(2, \mathbb{R})$ (the universal covering of the special linear group). For details, see [5]. Recall that we are assuming in this paper that X is non-compact, so in particular X cannot be isometric to a metric Lie group isomorphic to $SU(2)$.

We will use later the following result, which follows rather easily from the work of Peyerimhoff and Samiou [9]; see also Theorem 3.32 in [5] for a self-contained proof.

Proposition 2.6. *Suppose that X is isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric. Then, $\text{Ch}(X) = \text{trace}(A) = 2H(X)$. Furthermore, X is unimodular if and only if $\text{Ch}(X) = 0$.*

In particular, the equality $\text{Ch}(X) = 2H(X)$ in item (2) of Theorem 1.4 holds when X is isometric to a semidirect product with its canonical metric.

3 Isoperimetric domains when $\text{Ch}(X) = 0$.

Theorem 3.1. *Let X be a non-compact, simply connected, homogeneous three-manifold and let $\{\Omega_n\}_n$ be any sequence of isoperimetric domains in X with volumes tending to infinity. Then, the following statements are equivalent:*

- (A) $\text{Ch}(X) = 0$.
- (B) X is isometric either to the Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ of a 2-sphere of constant curvature $\kappa > 0$ with the real line, or to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some 2×2 real matrix A with trace zero, endowed with its canonical metric.
- (C) $\lim_{n \rightarrow \infty} \frac{\text{Area}(\partial\Omega_n)}{\text{Vol}(\Omega_n)} = 0$.
- (D) The mean curvatures H_n of $\partial\Omega_n$ are non-negative and satisfy $\lim_{n \rightarrow \infty} H_n = 0$.
- (E) Given any $R > 0$, $\lim_{n \rightarrow \infty} \frac{\text{Vol}(\Omega_n(R))}{\text{Vol}(\Omega_n)} = 0$, where $\Omega_n(R) = \{x \in \Omega_n \mid \text{dist}_X(x, \partial\Omega_n) < R\}$.

Furthermore, if any of the above conditions hold, then the sequence of radii of the Ω_n tends to infinity as $n \rightarrow \infty$.

Proof of Theorem 3.1. We first observe that if item (E) in Theorem 3.1 holds, then the last property in the statement of the theorem concerning the radii of the domains Ω_n also holds: otherwise, after passing to a subsequence there exists $C > 0$ such that for all

$x \in \Omega_n$ and for all $n \in \mathbb{N}$, $\text{dist}_X(x, \partial\Omega_n) \leq C$. This implies that $\Omega_n(R) = \Omega_n$ for all $R > C$, which contradicts the assumption in item (E).

We will prove the equivalence between the items in Theorem 3.1 in six steps.

Step 1: *If X is isometric to the Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for some $\kappa > 0$, then items (A), (B), (C), (D) and (E) in Theorem 3.1 hold.*

Proof of Step 1. Consider the smooth domains $\Omega_R = \mathbb{S}^2(\kappa) \times [0, R]$ with $R > 0$. Then,

$$\frac{\text{Area}(\partial\Omega_R)}{\text{Vol}(\Omega_R)} = \frac{2 \text{Area}(\mathbb{S}^2(\kappa))}{R \text{Area}(\mathbb{S}^2(\kappa))} \rightarrow 0 \quad \text{as } R \rightarrow \infty, \quad (3.1)$$

and thus, $\text{Ch}(X) = 0$ and item (A) of Theorem 3.1 holds. By item (iii) of Remark 2.1, (3.1) gives that items (C) and (E) of Theorem 3.1 hold. As the boundary of Ω_R is minimal for R sufficiently large, then item (D) of the theorem also holds. Item (B) holds by the assumption that X is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$. \square

Step 2: *Item (A) and item (B) in Theorem 3.1 are equivalent.*

Proof of Step 2. A result by Hoke [3] states that $\text{Ch}(G) = 0$ for a non-compact, simply connected Lie group G (of any dimension) with a left invariant metric if and only if G is unimodular and amenable. Therefore, Step 2 follows from Step 1, Proposition 2.6 and this result by Hoke, since $\widetilde{\text{SL}}(2, \mathbb{R})$ is not amenable. \square

Step 3: *Items (A) and (C) in Theorem 3.1 are equivalent.*

Proof of Step 3. This follows directly from Lemma 2.4. \square

Step 4: *Item (D) in Theorem 3.1 implies item (A).*

Proof of Step 4. Assume that item (D) in Theorem 3.1 holds. Arguing by contradiction, suppose that $\text{Ch}(X) > 0$. Choose $V_0 > 0$ sufficiently large so that for every isoperimetric domain $\Omega \subset X$ with volume at least V_0 , the mean curvature of $\partial\Omega$ is at most $\text{Ch}(X)/4$ (this V_0 exists by our hypothesis in item (D)). Hence given $a > 0$,

$$\begin{aligned} \frac{I(V_0 + a) - I(V_0)}{V_0 + a} &= \frac{1}{V_0 + a} \int_{V_0}^{V_0 + a} I'_+(t) dt \\ &\stackrel{(*)}{\leq} \frac{1}{V_0 + a} \int_{V_0}^{V_0 + a} 2 \frac{\text{Ch}(X)}{4} dt = \frac{a \text{Ch}(X)}{2(V_0 + a)} < \frac{\text{Ch}(X)}{2}, \end{aligned}$$

where we have used in $(*)$ our hypothesis on the mean curvature of isoperimetric domains with volume at least V_0 and property (II) stated just before Remark 2.1. Since $\frac{I(V_0 + a)}{V_0 + a} > \text{Ch}(X)$ by item (2) of Lemma 2.3, the inequalities

$$\text{Ch}(X) - \frac{I(V_0)}{V_0 + a} < \frac{I(V_0 + a) - I(V_0)}{V_0 + a} < \frac{\text{Ch}(X)}{2}.$$

cannot hold for a sufficiently large under our assumption that $\text{Ch}(X) > 0$. This contradiction proves $\text{Ch}(X) = 0$, which completes the proof of this step. \square

Step 5: Item (E) and item (A) in Theorem 3.1 are equivalent.

Proof of Step 5. We first check that item (E) implies item (A), so assume that item (E) holds.

As the volumes of the isoperimetric domains Ω_n are larger than some fixed $V > 0$, then their boundary surfaces $\partial\Omega_n$ have uniformly bounded second fundamental form (and hence also uniformly bounded constant mean curvatures H_n). We consider two possible cases.

If item (D) also holds, then by Step 4 we see that item (A) holds. Now assume that item (D) fails to hold in general. Thus, there is a sequence $\{\Omega'_n\}_n$ of isoperimetric domains with volumes tending to infinity, such that the mean curvatures H'_n of $\partial\Omega'_n$ are bounded away from zero. Since the sectional curvature of X is bounded from above, Theorem 3.5 in Meeks and Tinaglia [6] then ensures the existence of a positive number δ such that $\partial\Omega'_n$ has a regular δ -neighborhood in Ω'_n , i.e., the geodesic segments in Ω'_n of length δ and normal to $\partial\Omega'_n$ at one of their end points are embedded segments and they do not intersect each other, for each $n \in \mathbb{N}$. Since the surfaces $\partial\Omega'_n$ have uniformly bounded second fundamental forms and the absolute sectional curvature of X is bounded, there is a constant $c > 0$ such that $\text{Vol}(\Omega'_n(\delta)) \geq c \text{Area}(\partial\Omega'_n)$, for all $n \in \mathbb{N}$. Thus,

$$\text{Ch}(X) \leq \frac{\text{Area}(\partial\Omega'_n)}{\text{Vol}(\Omega'_n)} \leq \frac{1}{c} \frac{\text{Vol}(\Omega'_n(\delta))}{\text{Vol}(\Omega'_n)}, \quad \text{for all } n \in \mathbb{N}. \quad (3.2)$$

As we are assuming that item (E) in Theorem 3.1 holds, then the right-hand-side of (3.2) tends to zero as $n \rightarrow \infty$. Thus, $\text{Ch}(X) = 0$ and we conclude that item (E) implies item (A).

We next check that item (A) implies item (E). Fix $R > 0$ and note that given $n \in \mathbb{N}$ and $p \in \Omega_n(R)$, the closed metric ball in X

$$\overline{\mathbb{B}}(p, R) = \{x \in X \mid \text{dist}_X(x, p) \leq R\}$$

intersects the boundary $\partial\Omega_n$, where d denotes the distance function in X associated to its Riemannian metric. For $n \in \mathbb{N}$ fixed, consider the set \mathcal{A}_n whose elements are the pairwise disjoint collections $\{\overline{\mathbb{B}}(p_i, 2R) \mid i = 1, \dots, k\}$ for some collection of points $p_1, \dots, p_k \in \Omega_n(R)$ for some $k \in \mathbb{N}$. Note that \mathcal{A}_n is non-empty and it can be endowed with the partial order given by inclusion. Since $\overline{\Omega_n(R)}$ is compact, there exists a maximal element in \mathcal{A}_n for this partial order. In other words, for any fixed $n \in \mathbb{N}$, there exists a finite set $\{p_1, \dots, p_{k(n)}\} \subset \Omega_n(R)$ such that $\mathcal{C}_n = \{\overline{\mathbb{B}}(p_1, 2R), \dots, \overline{\mathbb{B}}(p_{k(n)}, 2R)\}$ is a maximal collection of pairwise disjoint closed balls with centers in $\Omega_n(R)$ and fixed radius

$2R$. This maximality implies that if $x \in \Omega_n(R)$, then there exists some $i \in \{1, \dots, k(n)\}$ such that $\overline{\mathbb{B}}(x, 2R) \cap \overline{\mathbb{B}}(p_i, 2R) \neq \emptyset$. Consequently, the triangle inequality gives that the collection

$$\mathcal{C}'_n = \{\overline{\mathbb{B}}(p_1, 4R), \dots, \overline{\mathbb{B}}(p_{k(n)}, 4R)\}$$

is a covering of $\Omega_n(R)$, and thus,

$$\text{Vol}(\Omega_n(R)) \leq \text{Vol} \left(\bigcup_{i=1}^{k(n)} \overline{\mathbb{B}}(p_i, 4R) \right) \quad (3.3)$$

$$\leq \sum_{i=1}^{k(n)} \text{Vol}(\overline{\mathbb{B}}(p_i, 4R)) = k(n) \text{Volume}(\overline{\mathbb{B}}(p_1, 4R)). \quad (3.4)$$

As the balls in \mathcal{C}_n are pairwise disjoint, we have

$$\text{Area}(\partial\Omega_n) \geq \sum_{i=1}^{k(n)} \text{Area}[\overline{\mathbb{B}}(p_i, 2R) \cap \partial\Omega_n]. \quad (3.5)$$

Since the norm of the second fundamental form of $\partial\Omega_n$ is uniformly bounded (independently of n), then there exists some $\tau \in (0, R)$ such that for all $p \in \partial\Omega_n$, the exponential map on the disk of radius τ in $T_p\partial\Omega_n$ is a quasi-isometry⁴, with constant depending only on X and the bound of the second fundamental form (and not depending on n). Then, since for each $i = 1, \dots, k(n)$, $\overline{\mathbb{B}}(p_i, R)$ intersects $\partial\Omega_n$ at some point q_i , then the intrinsic τ -disk centered at q_i has area not less than some number $\mu > 0$ not depending on n or q_i . Since the intrinsic distance dominates the extrinsic distance, then the triangle inequality implies that

$$\text{Area}[\overline{\mathbb{B}}(p_i, 2R) \cap \partial\Omega_n] \geq \mu, \quad \text{for all } i = 1, \dots, k(n) \text{ and } n \in \mathbb{N}. \quad (3.6)$$

Now, (3.3), (3.5) and (3.6) give for all $n \in \mathbb{N}$, the following inequalities:

$$\frac{\text{Area}(\partial\Omega_n)}{\text{Vol}(\Omega_n(R))} \geq \frac{\sum_{i=1}^{k(n)} \text{Area}[\overline{\mathbb{B}}(p_i, 2R) \cap \partial\Omega_n]}{k(n) \text{Volume}(\overline{\mathbb{B}}(p_1, 4R))} \geq \frac{\mu}{\text{Volume}(\overline{\mathbb{B}}(p_1, 4R))}. \quad (3.7)$$

Assume now that item (A) holds, and so item (C) also holds by Step 3, i.e.,

$$\lim_{n \rightarrow \infty} \frac{\text{Area}(\partial\Omega_n)}{\text{Vol}(\Omega_n)} = 0.$$

So the only way (3.7) can hold is that $\lim_{n \rightarrow \infty} \frac{\text{Vol}(\Omega_n(R))}{\text{Vol}(\Omega_n)} = 0$, which finishes the proof of Step 5. \square

⁴Recall that a *quasi-isometry* $f: (X, g) \rightarrow (Y, g')$ between Riemannian manifolds is a diffeomorphism satisfying $Cg \leq f^*g' \leq \frac{1}{C}g$ in X for some $C > 0$.

Step 6: Item (C) in Theorem 3.1 implies item (D).

Proof of Step 6. Assume that item (C) holds. Arguing by contradiction, suppose that item (D) of Theorem 3.1 fails to hold. Then Step 1 implies that X is not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, and so, by Remark 2.1, X is diffeomorphic to \mathbb{R}^3 and the boundaries of isoperimetric domains are connected with positive mean curvature. Therefore, the failure of item (D) to hold implies that there exists a sequence of isoperimetric domains Ω_n with volumes tending to infinity for which the mean curvatures H_n of $\partial\Omega_n$ satisfy $H_n \geq \beta$ for some number $\beta > 0$. As item (C) holds, then item (A) also holds by Step 3 and so $\text{Ch}(X) = 0$. By Step 5 we see that item (E) also holds, and thus the radii of the Ω_n diverge to infinity as $n \rightarrow \infty$ by the argument just before the statement of Step 1.

Since $\text{Ch}(X) = 0$, Lemma 2.5 implies that there exists a sequence $\{T_k\}_k \subset (0, \infty)$ diverging to infinity such that given a sequence $\{\Omega_k^1\}_k$ of isoperimetric domains in X with $\text{Volume}(\Omega_k^1) = T_k$, then the mean curvature H_k^1 of the boundary of Ω_k^1 satisfies $H_k^1 < 1/k$, for all $k \in \mathbb{N}$. Since each Ω_k^1 is compact and the radii of the Ω_n diverge to infinity as $n \rightarrow \infty$, then given $k \in \mathbb{N}$, there exists $n(k) \in \mathbb{N}$ such that, after an ambient isometry of X applied to $\Omega_{n(k)}$, we have $\Omega_k^1 \subset \Omega_{n(k)}$. As X is homogeneous, a simple application of the mean curvature comparison principle implies that $H_{n(k)} \leq H_k^1$ (move isometrically Ω_k^1 inside $\Omega_{n(k)}$ until the first time that their boundaries touch). This is a contradiction, since $\beta \leq H_{n(k)} \leq H_k^1 < 1/k$ for all k . This contradiction finishes the proof of Step 6. \square

Finally, note that Steps 1-6 above complete the proof of Theorem 3.1. \square

4 Simply connected homogeneous three-manifolds with $\text{Ch}(X) > 0$.

By the results in Section 2.2 and by Theorem 3.1, the condition $\text{Ch}(X) > 0$ is equivalent to the fact that X is isometric to a metric Lie group which is either a non-unimodular semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric, or $\widetilde{\text{SL}}(2, \mathbb{R})$ with a left invariant metric. Thus, *from this point on we will assume that X is identified with the related metric Lie group.* In particular given $a \in X$, the left translation $l_a: X \rightarrow X$ defined by $l_a(x) = ax$ (we will omit the group operation in X) is an isometry. Note that given a right invariant vector field K on X , its associated 1-parameter group of diffeomorphisms is the 1-parameter group of isometries $\{l_a \mid a \in \Gamma\}$, where $\Gamma \subset X$ is the 1-parameter subgroup of X defined by $\Gamma(0) = e$, $\Gamma'(0) = K(e)$ (in the sequel, e will denote the identity element of X). Therefore, K is a Killing vector field.

We next recall some properties of such a metric Lie group with $\text{Ch}(X) > 0$, that will be useful in later discussions. For detailed proofs of the properties stated below, see [5].

Case (A): X is a non-unimodular semidirect product.

Assume that $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

equipped with its canonical left invariant metric. An orthonormal left invariant frame for the canonical left invariant metric on X is

$$E_1(x, y, z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \quad E_2(x, y, z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z, \quad (4.1)$$

where $e^{zA} = (a_{ij}(z))_{i,j} = 1, 2$ and $\partial_x = \frac{\partial}{\partial x}, \partial_y, \partial_z$ is the usual parallelization of \mathbb{R}^3 . The Lie bracket is given by

$$[E_1, E_2] = 0, \quad [E_3, E_1] = aE_1 + cE_2, \quad [E_3, E_2] = bE_1 + dE_2.$$

In the natural coordinates $(x, y, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R}$, the canonical left invariant metric is given by

$$\begin{aligned} \langle, \rangle &= [a_{11}(-z)^2 + a_{21}(-z)^2] dx^2 + [a_{12}(-z)^2 + a_{22}(-z)^2] dy^2 + dz^2 \\ &+ [a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)] (dx \otimes dy + dy \otimes dx), \end{aligned} \quad (4.2)$$

and the Levi-Civita connection ∇ for the canonical left invariant metric is given by

$$\begin{array}{l|l|l} \nabla_{E_1} E_1 = a E_3 & \nabla_{E_1} E_2 = \frac{b+c}{2} E_3 & \nabla_{E_1} E_3 = -a E_1 - \frac{b+c}{2} E_2 \\ \nabla_{E_2} E_1 = \frac{b+c}{2} E_3 & \nabla_{E_2} E_2 = d E_3 & \nabla_{E_2} E_3 = -\frac{b+c}{2} E_1 - d E_2 \\ \nabla_{E_3} E_1 = \frac{c-b}{2} E_2 & \nabla_{E_3} E_2 = \frac{b-c}{2} E_1 & \nabla_{E_3} E_3 = 0. \end{array} \quad (4.3)$$

In particular (4.3) implies that the mean curvature of each leaf of the foliation $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ with respect to the unit normal vector field E_3 is the constant $H = \text{trace}(A)/2$, which equals $H(X) = \frac{1}{2}\text{Ch}(X)$ by Proposition 2.6.

Case (B): X is $\widetilde{\text{SL}}(2, \mathbb{R})$ equipped with a left invariant metric.

The 2×2 real matrices with determinant equal to 1 form the *special linear group* $\text{SL}(2, \mathbb{R})$, and represent the orientation-preserving linear transformations of \mathbb{R}^2 that preserve the oriented area. The quotient of $\text{SL}(2, \mathbb{R})$ modulo $\{\pm \text{identity}\}$ is the *projective special linear group* $\text{PSL}(2, \mathbb{R})$, isomorphic to the group of orientation-preserving isometries of the hyperbolic plane \mathbb{H}^2 , and naturally diffeomorphic to the unit tangent bundle of \mathbb{H}^2 . The fundamental groups of $\text{SL}(2, \mathbb{R})$, $\text{PSL}(2, \mathbb{R})$ are infinite cyclic, and the universal cover of both groups is the simply connected unimodular Lie group $\widetilde{\text{SL}}(2, \mathbb{R})$, which is the underlying algebraic structure of the homogeneous manifold X in this case (B).

$\text{PSL}(2, \mathbb{R})$ admits three types of 1-parameter subgroups, namely elliptic, parabolic and hyperbolic subgroups, that correspond to 1-parameter subgroups of Möbius transformations of the Poincaré disk with zero, one or two fixed points at the boundary at infinity

$\partial_\infty \mathbb{H}^2 = \mathbb{S}^1$, respectively. The Lie algebra of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ (and of $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{PSL}(2, \mathbb{R})$) is the linear space $\mathfrak{sl}(2, \mathbb{R})$ of 2×2 real matrices with trace zero, with the Lie bracket given by the commutator of matrices. The basis of $\mathfrak{sl}(2, \mathbb{R})$ described by the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4.4)$$

satisfies the Lie brackets relations

$$[E_1, E_2] = -2E_3, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = 2E_2.$$

The two-dimensional subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are non-commutative and form an \mathbb{S}^1 -family $\{\mathbb{H}_\mu^2 \mid \mu \in \mathbb{S}^1 = \partial_\infty \mathbb{H}^2\}$, where

$$\mathbb{H}_\mu^2 = \{\text{orientation-preserving isometries of } \mathbb{H}^2 \text{ which fix } \mu \in \partial_\infty \mathbb{H}^2\}. \quad (4.5)$$

Elements in \mathbb{H}_μ^2 are rotations around μ (parabolic) and translations along geodesics one of whose end points is μ (hyperbolic). The one-dimensional and two-dimensional subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ are the lifts via the covering map $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ of the corresponding one-dimensional and two-dimensional subgroups of $\mathrm{PSL}(2, \mathbb{R})$, and we will use accordingly the notation *elliptic*, *parabolic*, *hyperbolic* and \mathbb{H}_μ^2 for these connected lifted subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Under its left action, every 1-parameter subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generates a right invariant Killing vector field on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and we will also call these right invariant vector fields *elliptic*, *parabolic* and *hyperbolic*, accordingly to the nature of the related 1-parameter subgroups.

As mentioned earlier, there is a natural projection of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ to the hyperbolic plane. More specifically, there is a submersion

$$\Pi: \widetilde{\mathrm{SL}}(2, \mathbb{R}) \rightarrow \mathbb{H}^2, \quad (4.6)$$

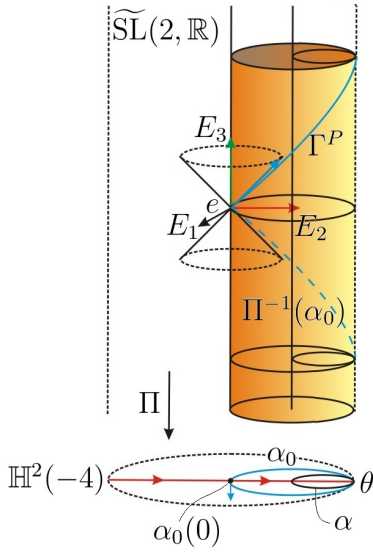
which is the composition of the covering map $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ with the natural projection from $\mathrm{PSL}(2, \mathbb{R})$ to \mathbb{H}^2 obtained after identifying $\mathrm{PSL}(2, \mathbb{R})$ with the unit tangent bundle of \mathbb{H}^2 .

With respect to the choice of basis (4.4) for the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, the center of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is an infinite cyclic subgroup contained in the integral curve $\Gamma^E \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the left invariant vector field E_3 in (4.4) that passes through the identity element e of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$; the image set of Γ^E is a 1-parameter elliptic subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

We next fix some notation that we will use in the remainder of this manuscript. Let Γ^H , Γ^E , Γ^P be the 1-parameter subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ given by

$$(\Gamma^H)'(0) = E_2(e), \quad (\Gamma^E)'(0) = E_3(e), \quad (\Gamma^P)'(0) = E_1(e) + E_3(e), \quad (4.7)$$

where E_1, E_2, E_3 are given by (4.4). Thus, Γ^H (resp. Γ^E, Γ^P) is a hyperbolic (resp. elliptic, parabolic) 1-parameter subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Let $\theta \in \partial_\infty \mathbb{H}^2 = \mathbb{S}^1$ be the end point of



The shaded surface in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is a horocylinder, inverse image by the projection Π of a punctured circle $\alpha_0 \subset \mathbb{H}^2$ tangent at a point in the boundary at infinity of \mathbb{H}^2 . $\Pi^{-1}(\alpha_0)$ is everywhere tangent to the parabolic right invariant vector field K^P on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generated by the 1-parameter parabolic subgroup Γ^P .

Figure 1:

the parameterized geodesic $\Pi(\Gamma^H)$ of \mathbb{H}^2 obtained by projecting Γ^H via the map Π given in (4.6), which is the end point of $\Pi(\Gamma^H([0, \infty)))$ at infinity in \mathbb{H}^2 . Let \mathbb{H}_θ^2 be the lift to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the two-dimensional subgroup of $\mathrm{PSL}(2, \mathbb{R})$ that consists of the isometries of \mathbb{H}^2 that fix θ . \mathbb{H}_θ^2 contains both 1-parameter subgroups Γ^P and Γ^H . Furthermore, Γ^P is the unique parabolic subgroup and the unique normal 1-parameter subgroup of \mathbb{H}_θ^2 . Left translations by elements in Γ^P (resp. in Γ^H) generate a right invariant parabolic vector field K^P (resp. hyperbolic vector field K^H) on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

Definition 4.1. Given a horocycle $\alpha \subset \mathbb{H}^2$, we call $\Pi^{-1}(\alpha) \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$ the *horocylinder* in X over α .

Let \mathcal{C} be the set of horocycles in \mathbb{H}^2 tangent to the $\theta \in \partial_\infty \mathbb{H}^2$ defined in the previous paragraph. For each $\alpha \in \mathcal{C}$, the horocylinder $\Pi^{-1}(\alpha) \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$ is everywhere tangent to the parabolic right invariant vector field K^P of X and $\Pi_*[(K^P)(e)] = \alpha'_0(0)$, where α_0 is the horocycle in \mathcal{C} that passes through $\Pi(e) \in \mathbb{H}^2$, parameterized appropriately and so that $\alpha_0(0) = \Pi(e)$, see Figure 1.

The family of left invariant metrics on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is three-parametric. Any such left invariant metric can be constructed by declaring that the left invariant vector fields E_1, E_2, E_3 in (4.4) are orthogonal with corresponding lengths being arbitrary positive numbers $\lambda_1, \lambda_2, \lambda_3 > 0$, respectively. Thus, we have:

Proposition 4.2. *The space of left invariant metrics on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ can be naturally parameterized by the open set $\mathcal{M} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \mid \lambda_i > 0, i = 1, 2, 3\}$, whereby we declare the ordered set $\{\lambda_1 E_1, \lambda_2 E_2, \lambda_3 E_3\}$ of left invariant vector fields on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ to be orthonormal. Henceforth, we will identify the space of left invariant metrics on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ with the set \mathcal{M} under the above correspondence.*

Among the left invariant metrics of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, we have a two-parameter subfamily contained in \mathcal{M} of those metrics that have an isometry group of dimension four; these special metrics $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{M}$ correspond to the case where $\lambda_1 = \lambda_2$. The generic case of a left invariant metric on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ has a three-dimensional group of isometries. For any left invariant metric on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the directions of E_1, E_2, E_3 can be proven to be principal directions for the Ricci tensor, and there exist orientation-preserving diffeomorphisms of order two around any of the integral curves of these three vector fields, which turn out to be isometries for every left invariant metric on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Hence, the 1-parameter subgroups Γ^H and Γ^E are geodesics in every metric described in Proposition 4.2, as they are the fixed point sets of rotational isometries in every such metric.

In the case that X is $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ equipped with a left invariant metric whose isometry group is four-dimensional, then after rescaling this left invariant metric we can consider $\Pi: X \rightarrow \mathbb{H}^2$ to be a Riemannian submersion with constant bundle curvature. In this case, the following property is well-known.

Lemma 4.3. *If X is $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ equipped with a left invariant metric whose isometry group is four-dimensional, then every horocylinder in X has constant mean curvature equal to the critical mean curvature⁵ $H(X)$ of X .*

In contrast to the statement of Lemma 4.3, if X is $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ equipped with a left invariant metric whose isometry group is three-dimensional, then for any horocycle α in \mathbb{H}^2 , the horocylinder $\Pi^{-1}(\alpha) \subset X$ does not have constant mean curvature.

5 Foliations by leaves of critical mean curvature in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

In this section we will use the notation developed in the previous section. We will **assume that X is $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ endowed with an arbitrary left invariant metric g** , and we will construct a topological product foliation of X by surfaces with constant mean curvature equal to $H(X)$. The existence and properties of this foliation will be important in the proof of Theorem 1.4 which we give in Section 6.

Definition 5.1. Let a_1 be a fixed element of $\Gamma^P - \{e\}$ and let a_2 be one of the two generators of the center of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ (in particular, $a_2 \in \Gamma^E - \{e\}$). Thus, the left translations l_{a_1}, l_{a_2} generate a discrete subgroup Δ of the isometry group of $X = (\widetilde{\mathrm{SL}}(2, \mathbb{R}), g)$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$ that acts properly and discontinuously on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. We define $W = \widetilde{\mathrm{SL}}(2, \mathbb{R})/\Delta$ and let

$$\pi_W : \widetilde{\mathrm{SL}}(2, \mathbb{R}) \rightarrow W \tag{5.1}$$

⁵This follows, for instance, from the fact that parallel horocylinders produce a foliation and are limits of spheres with constant mean curvature in such an X .

denote the corresponding quotient submersion. Note that the definition of W is independent of the left invariant metric g considered on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$; however, as Δ acts on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ by isometries of g , then W can be endowed with the quotient metric g_W so that

$$\pi_W : X \rightarrow (W, g_W) \quad (5.2)$$

becomes a local isometry and (W, g_W) is locally homogeneous.

The next two lemmas collect some properties that do not depend on the left invariant metric g on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Item (2) of Lemma 5.2 implies that W is diffeomorphic to the product of a torus with the real line, provided that the surface $\Sigma_1 \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$ in the statement of Lemma 5.2 exists. We will prove the existence of such surface Σ_1 in Lemma 5.3 below.

Lemma 5.2. *In the above situation, K^P is invariant under the left action of Δ , and so the left action of Γ^P on X induces a left action of Γ^P on W . Let $\Sigma_1 \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$ be a properly embedded surface invariant under the left action of Γ^P and under l_{a_2} . Then:*

1. *For all $h \in \Gamma^H$, the group Δ acts properly and discontinuously on $l_h(\Sigma_1)$ and the left action of Γ^P leaves invariant $l_h(\Sigma_1)$.*
2. *Suppose that each integral curve of K^H intersects Σ_1 transversely in a single point. Then*

$$\mathcal{F}(\Sigma_1, \Gamma^H) = \{l_h(\Sigma_1) \mid h \in \Gamma^H\}$$

is a product foliation of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and each of the leaves $l_h(\Sigma_1)$ of $\mathcal{F}(\Sigma_1, \Gamma^H)$ is a properly embedded topological plane invariant under the left action of Γ^P and under l_{a_2} . In particular, $\mathcal{F}(\Sigma_1, \Gamma^H)$ descends to the product foliation $\{l_h(\Sigma_1)/\Delta \mid h \in \Gamma^H\}$ of W , each of whose leaves $l_h(\Sigma_1)/\Delta$ is a torus invariant under the induced action of Γ^P on W .

Proof. K^P is l_{a_1} -invariant, since $a_1 \in \Gamma^P$ and the left action of Γ^P generates K^P . As a_2 lies in the center of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, then l_{a_2} coincides with the right translation $x \in \widetilde{\mathrm{SL}}(2, \mathbb{R}) \mapsto x a_2$. Since K^P is right invariant, then K^P is also l_{a_2} -invariant. Therefore, K^P is invariant under the left action of Δ , and the first sentence in the statement of the lemma follows. We will keep the notation K^P for the induced vector field⁶ on W , and Γ^P for the 1-parameter family of diffeomorphisms⁷ of W that generate K^P .

Now consider a properly embedded surface $\Sigma_1 \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$ invariant under the left action of Γ^P and under l_{a_2} . We first show that given $h \in \Gamma^H$ and $b \in \Delta$, the left translation by b leaves $l_h(\Sigma_1)$ invariant. As Δ is generated by a_1, a_2 , it suffices to consider the cases $b = a_1$ and $b = a_2$.

⁶Note that the induced vector field K^P on W is a Killing field for the quotient metric g_W defined by (5.2), independently on the left invariant metric g on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

⁷These are isometries of (W, g_W) .

(I) Assume $b = a_2$. Since a_2 lies in the center of $\widetilde{\text{SL}}(2, \mathbb{R})$, then $l_{a_2}(l_h(\Sigma_1)) = l_h(l_{a_2}(\Sigma_1)) \subset l_h(\Sigma_1)$, because $l_{a_2}(\Sigma_1) \subset \Sigma_1$.

(II) Suppose $b = a_1$. Since Γ^P is a normal subgroup of \mathbb{H}_θ^2 and $\Gamma^H \subset \mathbb{H}_\theta^2$, then $a_1 h = h a'_1$ for some $a'_1 \in \Gamma^P$, and thus $l_{a_1}(l_h(\Sigma_1)) = l_h(l_{a'_1}(\Sigma_1)) \subset l_h(\Sigma_1)$, because Σ_1 is invariant under the left translation by every element in Γ^P .

This proves that Δ leaves $l_h(\Sigma_1)$ invariant. The property that Δ acts properly and discontinuously on $l_h(\Sigma_1)$ is obvious. That the left action of Γ^P leaves $l_h(\Sigma_1)$ invariant follows from the previous proof in (II), taking into account that a_1 can be chosen to be any element in $\Gamma^P - \{e\}$. Now item (1) of the lemma is proved.

Next suppose that each integral curve of K^H intersects Σ_1 in a single point (in particular, Σ_1 is a properly embedded topological plane). The surfaces $l_h(\Sigma_1)$ with $h \in \Gamma^H$ are then pairwise disjoint, properly embedded topological planes in $\widetilde{\text{SL}}(2, \mathbb{R})$ that define a product foliation $\mathcal{F}(\Sigma_1, \Gamma^H)$ of $\widetilde{\text{SL}}(2, \mathbb{R})$, and item (1) implies that each leaf $l_h(\Sigma_1)$ of $\mathcal{F}(\Sigma_1, \Gamma^H)$ gives rise to a properly embedded quotient surface $l_h(\Sigma_1)/\Delta \subset W$ which is invariant under the induced action of Γ^P on W . Since the fundamental group of $l_h(\Sigma_1)$ is trivial, then the fundamental group of $l_h(\Sigma_1)/\Delta$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and so $l_h(\Sigma_1)/\Delta$ is a torus. Finally, when h varies in Γ^H , the related tori $l_h(\Sigma_1)/\Delta$ are pairwise disjoint, thereby defining a product foliation of W . \square

Recall from (4.7) in the previous section that we parameterized the 1-parameter subgroup Γ^H by a group homomorphism

$$t \in \mathbb{R} \mapsto h(t) = \Gamma^H(t) \quad \text{with } (\Gamma^H)'(0) = E_2(e). \quad (5.3)$$

In particular, for every left invariant metric g on $\widetilde{\text{SL}}(2, \mathbb{R})$, the parameterized curve $t \mapsto h(t) = \Gamma^H(t)$ is an embedded geodesic (because its velocity vector $h'(t)$ has constant length λ_2 if g corresponds to a triple $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{R}^+)^3$ by the correspondence of Proposition 4.2, and its trace Γ^H is the set of fixed points of an order-two, orientation-preserving isometry of $X = (\widetilde{\text{SL}}(2, \mathbb{R}), g)$ around Γ^H). In the sequel we will identify Γ^H with \mathbb{R} through the parameterization h . When $t \rightarrow +\infty$, we produce an end point $\theta \in \partial_\infty \mathbb{H}^2$ of the projection of $\Gamma^H = h(\mathbb{R})$ through the map $\Pi: \widetilde{\text{SL}}(2, \mathbb{R}) \rightarrow \mathbb{H}^2$ defined in (4.6). Associated to θ we have the set \mathcal{C} of horocycles in \mathbb{H}^2 tangent to θ at infinity. Inside \mathcal{C} we distinguished the horocycle α_0 that passes through $\Pi(e) \in \mathbb{H}^2$, parameterized appropriately so that $\alpha_0(0) = \Pi(e)$. We called the lifted surface $\Pi^{-1}(\alpha_0)$ the *horocylinder* over α_0 ; see Figure 1.

Lemma 5.3. *The horocylinder $\Sigma_0 = \Pi^{-1}(\alpha_0)$ satisfies the hypotheses of the surface Σ_1 in Lemma 5.2. Hence, the product foliation of X*

$$\mathcal{F}_0 = \mathcal{F}(\Sigma_0, \Gamma^H) = \{l_h(\Sigma_0) \mid h \in \Gamma^H\} \quad (5.4)$$

descends to a product foliation \mathcal{F}_0/Δ of W by the tori $l_t(\Sigma_0)/\Delta$, $t \in \mathbb{R} = \Gamma^H$, each of which is invariant under the induced action of Γ^P on W , and W is diffeomorphic to the product of a torus with \mathbb{R} .

Proof. Consider the auxiliary left invariant metric g_0 on $\widetilde{\text{SL}}(2, \mathbb{R})$ that makes the basis in (4.4) orthonormal. Thus, the isometry group of $X_0 = (\widetilde{\text{SL}}(2, \mathbb{R}), g_0)$ is four-dimensional and the projection $\Pi: X_0 \rightarrow \mathbb{H}^2$ given by (4.6) is, after rescaling the left invariant metric, a Riemannian submersion, as explained at the end of Section 4. It is worth remembering that X_0 is isometric to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with

$$A = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}, \quad (5.5)$$

endowed with its canonical left invariant metric; to see why this property holds, we can apply part (2) of Theorem 2.14 of [5] to conclude that X_0 is isometric to $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ where $A(b) = \begin{pmatrix} 2 & 0 \\ 2b & 0 \end{pmatrix}$; to see that $b = 1$, simply observe that the eigenvalues of the Ricci tensor of X_0 are -6 (double) and 2 (simple), while the eigenvalues of the Ricci tensor of a semidirect product with its canonical metric are given by equation (2.23) in [5]. Equality in both collections of Ricci eigenvalues easily lead to the desired property that $b = 1$. The Riemannian three-manifold X_0 is commonly referred to as an $\mathbb{E}(\kappa, \tau)$ -space with $\kappa = -4$ and $\tau^2 = 1$; to be precise, $\Pi: X_0 \rightarrow \mathbb{H}^2$ is a Riemannian submersion, where the usual metric on \mathbb{H}^2 has been scaled so that it has sectional curvature -4 .

The reader should be aware that in spite of the fact that X_0 is isometric to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric, the group structure on X_0 (that is, the one of $\widetilde{\text{SL}}(2, \mathbb{R})$) is not isomorphic to the one given by (2.5) in $\mathbb{R}^2 \rtimes_A \mathbb{R}$, as follows from the fact that $\widetilde{\text{SL}}(2, \mathbb{R})$ is unimodular while $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is non-unimodular with the matrix A above. Nevertheless, the non-isomorphic Lie groups $\widetilde{\text{SL}}(2, \mathbb{R})$ and $\mathbb{R}^2 \rtimes_A \mathbb{R}$ can be considered as three-dimensional subgroups of the four-dimensional isometry group $\text{Iso}(X_0)$ of X_0 , both acting by left translation on $\text{Iso}(X_0)$ with the same identity element equal to $1_X \in \text{Iso}(X_0)$. In this setting, Corollary 3.19 in [5] ensures that the connected component of $[\widetilde{\text{SL}}(2, \mathbb{R}) \cap (\mathbb{R}^2 \rtimes_A \mathbb{R})] \subset \text{Iso}(X_0)$ passing through 1_X is the two-dimensional subgroup \mathcal{H} of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ given by

$$\mathcal{H} = \{(x, x, z) \mid x, z \in \mathbb{R}\},$$

and the entire intersection $[\widetilde{\text{SL}}(2, \mathbb{R}) \cap (\mathbb{R}^2 \rtimes_A \mathbb{R})] \subset \text{Iso}(X_0)$ is generated by \mathcal{H} and the center of $\widetilde{\text{SL}}(2, \mathbb{R})$. Viewed inside $\widetilde{\text{SL}}(2, \mathbb{R})$, \mathcal{H} corresponds to one of the non-commutative subgroups \mathbb{H}_μ^2 given by (4.5) for some $\mu \in \partial_\infty \mathbb{H}^2$. After conjugating the embedding of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ into $\text{Iso}(X_0)$ by an appropriate isometry of X_0 which is a rotation around Γ^E , we can identify \mathcal{H} inside $\widetilde{\text{SL}}(2, \mathbb{R})$ with the two-dimensional subgroup \mathbb{H}_θ^2 defined just before Definition 4.1. Therefore, geometric and algebraic objects inside $\mathcal{H} \subset \mathbb{R}^2 \rtimes_A \mathbb{R}$ have their

counterparts inside $\mathbb{H}_\theta^2 \subset \widetilde{\text{SL}}(2, \mathbb{R})$; for instance, the unique parabolic 1-parameter normal subgroup $\{(x, x, 0) \mid x \in \mathbb{R}\}$ of \mathcal{H} corresponds to Γ^P inside \mathbb{H}_θ^2 , and the hyperbolic 1-parameter subgroup $\{(0, 0, z) \mid z \in \mathbb{R}\}$ (which is a unit speed geodesic of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ parameterized by $z \in \mathbb{R}$) corresponds to the hyperbolic 1-parameter subgroup Γ^H inside \mathbb{H}_θ^2 .

Under the above isometric identification of $\widetilde{\text{SL}}(2, \mathbb{R})$ with $\mathbb{R}^2 \rtimes_A \mathbb{R}$, the horocylinder $\Sigma_0 \subset \widetilde{\text{SL}}(2, \mathbb{R})$, which has constant mean curvature in X_0 , corresponds to the horizontal plane $\mathbb{R}^2 \rtimes_A \{0\}$ (note that $\mathbb{R}^2 \rtimes_A \{0\}$ is a subgroup of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ but Σ_0 is not a subgroup of $\widetilde{\text{SL}}(2, \mathbb{R})$), and the center of $\widetilde{\text{SL}}(2, \mathbb{R})$, which is contained in Σ_0 , can be viewed as a discrete subset of $\mathbb{R}^2 \rtimes_A \{0\} \subset \mathbb{R}^2 \rtimes_A \mathbb{R}$. From here is not difficult to conclude that the $(\mathbb{Z} \times \mathbb{Z})$ -subgroup Δ of $\widetilde{\text{SL}}(2, \mathbb{R})$ corresponds to a $(\mathbb{Z} \times \mathbb{Z})$ -lattice $\tilde{\Delta}$ inside the two-dimensional subgroup $\mathbb{R}^2 \rtimes_A \{0\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. As a consequence, the quotient space $W = \widetilde{\text{SL}}(2, \mathbb{R})/\Delta$ is diffeomorphic to $(\mathbb{R}^2 \rtimes_A \mathbb{R})/\tilde{\Delta}$, which is diffeomorphic to the product of the two-torus $(\mathbb{R}^2 \rtimes_A \{0\})/\tilde{\Delta}$ with the real line $\{(0, 0, z) \mid z \in \mathbb{R}\}$. Now the properties in the statement of the lemma follow easily from the corresponding properties in $\mathbb{R}^2 \rtimes_A \mathbb{R}$; for instance, the foliation $\mathcal{F}_0 = \mathcal{F}(\Sigma_0, \Gamma^H)$ corresponds to $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$. This completes the proof. \square

We will use the notation introduced in the proof of the previous lemma to make an observation that will be useful later.

Lemma 5.4. *Consider the left invariant metric g_0 on $\widetilde{\text{SL}}(2, \mathbb{R})$ that makes the basis in (4.4) orthonormal. Let $N_{\mathcal{F}_0}$ be the unit normal vector field to the foliation \mathcal{F}_0 of $X_0 = (\widetilde{\text{SL}}(2, \mathbb{R}), g_0)$ by flat horocylinders defined in (5.4). Then:*

$$N_{\mathcal{F}_0} = K^H + f K^P \quad (5.6)$$

for some smooth function $f: \widetilde{\text{SL}}(2, \mathbb{R}) \rightarrow \mathbb{R}$.

Proof. As described in the proof of Lemma 5.3, Γ^H and $\Gamma^P \subset \widetilde{\text{SL}}(2, \mathbb{R})$ correspond, respectively, to the 1-parameter subgroups $\{(0, 0, z) \mid z \in \mathbb{R}\}$ and $\{(x, x, 0) \mid x \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Keeping the same notation as in the proof of Lemma 5.3, let

$$F_3 = 2x \partial_x + 2x \partial_y + \partial_z, \quad \hat{K}^P = \partial_x + \partial_y$$

be the right invariant vector fields on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ generated by the right actions of $\{(0, 0, z) \mid z \in \mathbb{R}\}$ and $\{(x, x, 0) \mid x \in \mathbb{R}\}$, respectively. Then we have

$$K^H = F_3 = 2x \hat{K}^P + \partial_z.$$

As the unit normal vector field to the foliation $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is ∂_z , then the last equality implies that (5.6) holds with $f = 2x: \mathbb{R}^2 \rtimes_A \mathbb{R} \rightarrow \mathbb{R}$, which can be considered to be a smooth function on $\widetilde{\text{SL}}(2, \mathbb{R})$. \square

We now collect some properties involving any prescribed left invariant metric g on $\widetilde{\text{SL}}(2, \mathbb{R})$ and its locally homogeneous quotient metric g_W on W .

Lemma 5.5. *Let E_W be the end of W that contains the proper arc $\pi_W(\Gamma^H[0, \infty))$, where Γ^H is parameterized as a subgroup by $h: \mathbb{R} \rightarrow \Gamma^H$ defined in (5.3). Given a left invariant metric g on $\widetilde{\text{SL}}(2, \mathbb{R})$, the following properties hold:*

1. *The locally homogenous manifold (W, g_W) has infinite volume and the volume with respect to g_W of every end representative of E_W is finite.*
2. *If $\Sigma_1 \subset X$ is any surface satisfying the hypotheses of Lemma 5.2-(2), then the area function of the related quotient tori,*

$$t \in \mathbb{R} \mapsto A(t) = \text{Area}(l_{h(t)}(\Sigma_1)/\Delta, g_W)$$

is exponentially decreasing as $t \rightarrow +\infty$. Furthermore, an end representative of E_W can be chosen to be $\bigcup_{t \in [0, \infty)} [l_{h(t)}(\Sigma_1)/\Delta]$.

Proof. Since any two left invariant metrics on a metric Lie group are quasi isometric by the identity mapping, it suffices to prove the result for any particular left invariant metric on $\widetilde{\text{SL}}(2, \mathbb{R})$. So, choose the metric $g = g_0$ on $\widetilde{\text{SL}}(2, \mathbb{R})$ to be the one that makes the basis in (4.4) orthonormal.

As explained in the proof of Lemma 5.3, $X_0 = (\widetilde{\text{SL}}(2, \mathbb{R}), g_0)$ is isometric to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric (A is the matrix given by (5.5)) and the foliation $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, all whose leaves have the same constant mean curvature, corresponds to the foliation \mathcal{F}_0 of X_0 by parallel horocylinders defined in (5.4). Recall from the proof of Lemma 5.4 that the unit vector field ∂_z in the model $\mathbb{R}^2 \rtimes_A \mathbb{R}$ of X_0 corresponds to the unit normal field to the leaves of $\mathcal{F}(\Sigma_0, \Gamma^H)$.

By formula (3.51) in [5], the volume element associated to the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is

$$d\text{Vol} = e^{-z \text{trace}(A)} dx \wedge dy \wedge dz. \quad (5.7)$$

Equation (4.2) implies that ∂_z has constant length 1 and is everywhere orthogonal to the vectors fields ∂_x, ∂_y , whose lengths and inner product are given by

$$\|\partial_x\|^2 = e^{-4z} + (e^{-2z} - 1)^2, \quad \|\partial_y\| = 1, \quad \langle \partial_x, \partial_y \rangle = e^{-2z} - 1. \quad (5.8)$$

Now consider the $(\mathbb{Z} \times \mathbb{Z})$ -lattice $\widetilde{\Delta} \subset \mathbb{R}^2 \rtimes_A \{0\}$ corresponding to the subgroup $\Delta \subset \widetilde{\text{SL}}(2, \mathbb{R})$ through the isometry between X_0 and $\mathbb{R}^2 \rtimes_A \mathbb{R}$ (see the end of the proof of Lemma 5.3). Equation (5.7) implies that the end of $(\mathbb{R}^2 \rtimes_A \mathbb{R})/\widetilde{\Delta}$ in which the geodesic arc $\{(0, 0)\} \times [0, \infty)$ lies has finite volume with respect to the quotient of the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ (and the other end has infinite volume), hence the same properties hold

for (W, g_W) . It also follows that the area element of the plane $\mathbb{R}^2 \rtimes_A \{z\}$ has the form $d\text{Area} = e^{-z \text{trace}(A)} dx \wedge dy$, which implies that the areas of the tori $(\mathbb{R}^2 \rtimes_A \{z\})/\tilde{\Delta}$ in the related foliation of $(\mathbb{R}^2 \rtimes_A \mathbb{R})/\tilde{\Delta}$ decrease exponentially as a function of $z \rightarrow +\infty$. In fact, if $\Sigma_1 \subset X$ is a surface satisfying the hypotheses in item (2) of the lemma, then Σ_1 can be considered to be a doubly-periodic graph over $\mathbb{R}^2 \rtimes_A \{0\}$, from where we directly deduce that the $(g_0)_W$ -area of $l_{h(t)}\Sigma/\Delta$ decreases exponentially as $t \rightarrow +\infty$ (since the tori $(\mathbb{R}^2/\Delta) \rtimes_A \{t\}$ have the same property). This completes the proof of the lemma. \square

The main result of this section is the following one:

Proposition 5.6. *Let X be $\widetilde{\text{SL}}(2, \mathbb{R})$ endowed with a left invariant metric g . Then, $2H(X) = \text{Ch}(X)$. Furthermore, there exists a properly embedded surface $\Sigma \subset X$ with constant mean curvature $H(X)$ such that the following properties hold.*

- (A) $\Gamma^P \subset \Sigma$ and K^P is everywhere tangent to Σ (equivalently, Σ is invariant under the left translation by every element in Γ^P and $e \in \Sigma$).
- (B) Σ intersects each of the integral curves of K^H transversely in a single point. Equivalently, Σ is an entire graph with respect to the Killing vector field K^H on X generated by Γ^H . In particular, Σ is a topological plane and

$$\mathcal{F} = \mathcal{F}(\Sigma, \Gamma^H) = \{l_h(\Sigma) \mid h \in \Gamma^H\}$$

is a product foliation of X , all whose leaves have constant mean curvature $H(X)$.

- (C) Σ is invariant under the left translation l_{a_2} . Furthermore, the $(\mathbb{Z} \times \mathbb{Z})$ -subgroup Δ of isometries of X appearing in Definition 5.1 acts properly and discontinuously on Σ , thereby defining a quotient surface Σ/Δ in $(W = \widetilde{\text{SL}}(2, \mathbb{R})/\Delta, g_W)$.
- (D) Each leaf of the foliation \mathcal{F} is invariant under Δ , and \mathcal{F} descends to a product quotient foliation \mathcal{F}/Δ of (W, g_W) by tori with constant mean curvature $H(X)$.
- (E) Given $T \in \mathbb{R}$, consider the domain

$$\mathcal{D}(T) = \bigcup_{t=T}^{\infty} [l_{h(t)}(\Sigma)/\Delta], \quad (5.9)$$

which is an end representative of the end E_W of (W, g_W) with finite volume. Then, $\mathcal{D}(T)$ is the unique solution to the isoperimetric problem in (W, g_W) for its (finite) value of the enclosed volume. Moreover, the mean curvature vector of $\partial\mathcal{D}(T)$ points into $\mathcal{D}(T)$.

Before proving the above proposition, we state and prove the following key lemma whose proof will occupy several pages of this section.

Lemma 5.7. *Let X be $\widetilde{\text{SL}}(2, \mathbb{R})$ endowed with a left invariant metric. Suppose $\Sigma \subset X$ is a properly embedded surface of constant mean curvature $H \geq 0$ that satisfies conditions (A), (B) and (C) in Proposition 5.6. Then, $2H = \text{Ch}(X) = 2H(X)$ and Σ satisfies the remaining properties (D) and (E) in Proposition 5.6.*

Proof. By Lemma 5.2, Δ acts properly and discontinuously on every leaf of the product foliation $\mathcal{F} = \mathcal{F}(\Sigma, \Gamma^H)$, and \mathcal{F} descends to a quotient product foliation $\mathcal{F}/\Delta = \{l_{h(t)}(\Sigma)/\Delta \mid t \in \mathbb{R}\}$ of W by tori. In particular, W is diffeomorphic to $(\Sigma/\Delta) \times \mathbb{R}$. As Σ has constant mean curvature H in X and $\Delta \subset \text{Iso}(X)$, then Σ/Δ has constant mean curvature H in (W, g_W) and the same holds for all the leaves of \mathcal{F}/Δ because these leaves are all isometric by an ambient isometry. So, in order to show that Σ satisfies condition (D) of Proposition 5.6, it remains to prove that $H = H(X) = \frac{1}{2}\text{Ch}(X)$, which we do next.

By Lemma 5.5, the end E_W containing the proper arc $\pi_W(\Gamma^H[0, \infty))$ has finite volume with respect to g_W . Recall that we defined in (5.9) the end representative $\mathcal{D}(T)$ of E_W , for each $T \in \mathbb{R}$. It is worth reparameterizing $T \mapsto \mathcal{D}(T)$ by its enclosed finite volume with respect to g_W : as $T \in \mathbb{R} \mapsto \text{Vol}(\mathcal{D}(T), g_W)$ is strictly decreasing, we can define for each $V > 0$ the end representative

$$\Omega(V) := \mathcal{D}(T(V)) = \bigcup_{t=T(V)}^{\infty} [l_{h(t)}(\Sigma)/\Delta],$$

of E_W where $T(V)$ is uniquely defined by the equality $\text{Vol}(\Omega(V), g_W) = V$.

Assertion 5.8. *In the above situation, $H \neq 0$ and given $V > 0$, the mean curvature vector of $\partial\Omega(V)$ points into $\Omega(V)$. Equivalently, for every $T \in \mathbb{R}$, the mean curvature vector of $\partial\mathcal{D}(T)$ points into $\mathcal{D}(T)$.*

Proof. We will apply the Divergence Theorem to the unit normal field $N_{\mathcal{F}/\Delta}$ of the foliation \mathcal{F}/Δ (with respect to the metric g_W). As the leaves of \mathcal{F}/Δ are the quotient tori of left translations of Σ by elements $h \in \Gamma^H$, then the unit normal vector $N_{\mathcal{F}}$ to \mathcal{F} satisfies

$$N_{\mathcal{F}}(l_h(q)) = (l_h)_*(N_{\Sigma}(q)), \quad \text{for every } q \in \Sigma, h \in \Gamma^H,$$

where N_{Σ} stands for the unit normal vector to Σ (we will assume without loss of generality that N_{Σ} is the unit normal for which the mean curvature of Σ is $H \geq 0$). Furthermore, the divergence of $N_{\mathcal{F}}$ (resp. of $N_{\mathcal{F}/\Delta}$) with respect to g (resp. to g_W) is equal to the negative of twice the mean curvature H of the leaves of \mathcal{F} (resp. of \mathcal{F}/Δ). Since $\Omega(V)$ has finite volume and $\text{div}_W(N_{\mathcal{F}/\Delta})$ has a fixed sign, the Divergence Theorem gives

$$-2H V = -2H \text{Vol}(\Omega(V), g_W) = \int_{\Omega(V)} \text{div}_W(N_{\mathcal{F}/\Delta}) = \int_{\partial\Omega(V)} g_W(N_{\mathcal{F}/\Delta}, N_{\partial\Omega(V)}),$$

where $N_{\partial\Omega(V)}$ is the outward pointing unit normal of $\partial\Omega(V)$. Since $\partial\Omega(V) = l_{h(T(V))}(\Sigma)/\Delta$, then $(N_{\mathcal{F}/\Delta})|_{\partial\Omega(V)} = \varepsilon N_{\partial\Omega(V)}$ where $\varepsilon = \pm 1$ and the last equation reads $-2H V = \varepsilon \text{Area}(l_{h(T(V))}(\Sigma)/\Delta, g_W)$. In particular, $H > 0$ and $\varepsilon = -1$. Finally, the mean curvature vector of $\partial\Omega(V)$ is $H(N_{\mathcal{F}/\Delta})|_{\partial\Omega(V)} = -H N_{\partial\Omega(V)}$, which points into $\Omega(V)$. \square

Given $q \in X$, let $\gamma_q: \mathbb{R} \rightarrow X$ be the integral curve of $N_{\mathcal{F}}$ passing through q , i.e. $\gamma_q(0) = q$, $\gamma'_q(u) = N_{\mathcal{F}}(\gamma_q(u))$, for all $u \in \mathbb{R}$ (note that γ_q is defined for every value of $u \in \mathbb{R}$ as X is complete and $N_{\mathcal{F}}$ is bounded). Let $\phi_u: X \rightarrow X$, $\phi_u(q) = \gamma_q(u)$, for all $u \in \mathbb{R}$. Thus, $\{\phi_u\}_{u \in \mathbb{R}}$ is the 1-parameter group of diffeomorphisms of X generated by $N_{\mathcal{F}}$.

Assertion 5.9. *There exist constants $C_1, C_2 > 1$ such that:*

1. *For any unit tangent vector $v_p \in T_p X$ at a point $p \in X$ and $u \in [0, \infty]$, $\frac{1}{C_1 e^u} \leq |(\phi_u)_*(v_p)| \leq C_1 e^u$.*
2. *Recall that Γ^H was considered to be parameterized by $h(t) \in \Gamma^H$, $t \in \mathbb{R}$ (equation (5.3)). Then for any point $q \in \Sigma$ and $t_0 \geq 1$, it holds that $\gamma_q(C_2 t_0)$ lies in the set $\bigcup_{t \in [t_0, \infty)} l_{h(t)}(\Sigma)$.*

Proof. Recall that Σ is invariant under the left action of Δ , with quotient a torus Σ/Δ . Since left translations by elements in $\Delta \cup \Gamma^H$ leave invariant \mathcal{F} , then $N_{\mathcal{F}}$ is also invariant under left translations by elements in $\Delta \cup \Gamma^H$. This property together with the invariance of Σ under the left action of Δ imply that $|(\phi_u)_*(v_p)|$ is uniformly bounded and bounded away from zero independently of $u \in [0, 1]$, $p \in X$ and $v_p \in T_p X$ with $|v_p| = 1$. A straightforward iteration argument in the variable u gives the estimate in item (1) of Assertion 5.9.

To prove item (2), consider the function $\tilde{f}: X \rightarrow \mathbb{R}$ defined by

$$\tilde{f}(p) = t \quad \text{if} \quad p \in l_{h(t)}(\Sigma). \quad (5.10)$$

Clearly, \tilde{f} is invariant under the left action of Δ as each left translate of Σ has the same property. Take a point $p \in X$ and call $t = \tilde{f}(p)$. Given $s \in \mathbb{R}$, $l_{h(s)}(p) \in (l_{h(s)} \circ l_{h(t)})(\Sigma) = l_{h(s+t)}(\Sigma)$ because $t \mapsto h(t)$ is a group homomorphism. Thus,

$$\tilde{f} \circ l_{h(s)} = s + \tilde{f}, \quad \forall s \in \mathbb{R}. \quad (5.11)$$

A simple consequence of (5.11) is that the gradient $\nabla \tilde{f}$ of \tilde{f} with respect to the metric g satisfies $(\nabla \tilde{f}) \circ l_{h(s)} = (l_{h(s)})_*(\nabla \tilde{f})$, i.e., $\nabla \tilde{f}$ is invariant under the left action of Γ^H . $\nabla \tilde{f}$ is also invariant under the left action of Δ , as \tilde{f} has the same property. Observe that $\nabla \tilde{f}$ is orthogonal to $l_{h(t)}(\Sigma)$, hence

$$\nabla \tilde{f} = |\nabla \tilde{f}| N_{\mathcal{F}}. \quad (5.12)$$

Another simple consequence of (5.11) is that $\nabla \tilde{f}$ is nowhere zero on X (take the derivative with respect to s).

As $\nabla \tilde{f}$ descends without zeros to the torus $[l_{h(t)}(\Sigma)]/\Delta$ for any t , then $\nabla \tilde{f}$ is bounded by above and below by some positive constants in every leaf $l_{h(t)}(\Sigma)$ of \mathcal{F} . Since $\nabla \tilde{f}$ is also invariant under the action of Γ^H , we deduce that

$$C \leq |\nabla \tilde{f}| \leq 1/C \quad \text{in } X, \quad \text{for some } C \in (0, 1). \quad (5.13)$$

Let $(q, u) \in \Sigma \times [0, \infty) \mapsto t = t(q, u)$ be the smooth function such that $\gamma_q(u) \in l_{h(t)}(\Sigma)$. Then, $t(q, u) = \tilde{f}(\gamma_q(u))$ and

$$\begin{aligned} \frac{\partial t}{\partial u}(q, u) &= \frac{d}{du}(\tilde{f} \circ \gamma_q)(u) = g(\nabla \tilde{f}, \gamma'_q(u))(\gamma_q(u)) \\ &= g(\nabla \tilde{f}, N_{\mathcal{F}})(\gamma_q(u)) = |\nabla \tilde{f}|(\gamma_q(u)) \geq C; \end{aligned}$$

hence after integration with respect to u ,

$$t(q, u) \geq Cu - t(q, 0) = Cu, \quad \text{for all } u \geq 0, q \in \Sigma. \quad (5.14)$$

Now define $C_2 = 1/C$. Given $t_0 \geq 1$, $q \in \Sigma$, taking $u = C_2 t_0$ in (5.14) we have $t(q, C_2 t_0) \geq t_0$, which means that $\gamma_q(C_2 t_0) \in \bigcup_{t \in [t_0, \infty)} l_{h(t)}(\Sigma)$, as desired. This finishes the proof of Assertion 5.9. \square

The fact that Σ is invariant under the left action of Δ (by isometries of g) clearly implies the following statement.

Assertion 5.10. *Let σ_1, σ_2 be least-length simple closed geodesics in the torus Σ/Δ , that can be considered to be generators of the first homology group of Σ/Δ . Let $L_i > 0$ be the length of σ_i , $i = 1, 2$. For each $n \in \mathbb{N}$, let $F(n) \subset \Sigma$ be a 'square' collection of n^2 adjacent fundamental domains for the action of the group Δ , so that $\partial F(n)$ consists of four geodesic arcs in Σ , each of which covers n times one of the geodesics σ_1, σ_2 . Then,*

$$\text{Length}(\partial F(n), g) = C_3 n, \quad \text{Area}(F(n), g) = C_4 n^2, \quad (5.15)$$

where $C_3 = \text{Length}(\partial F(1)) = 2(L_1 + L_2)$, and $C_4 = \text{Area}(F(1))$.

Assertion 5.11. *In the above situation, $H = H(X) = \frac{1}{2}\text{Ch}(X)$ (and thus, Σ satisfies condition (D) of Proposition 5.6).*

Proof. Given a compact, orientable smooth surface S immersed in X , the existence of the foliation $\mathcal{F} = \mathcal{F}(\Sigma, \Gamma^H)$ of X by surfaces of constant mean curvature $H > 0$ and an elementary comparison argument for the mean curvature shows that the maximum of the absolute mean curvature function of S (with respect to the metric g) is at least H . Then, by definition of critical mean curvature we have $H \leq H(X)$. By Lemma 2.5, we

have $H(X) \leq \frac{1}{2}\text{Ch}(X)$; hence to finish the proof of the Assertion it suffices to show that $\text{Ch}(X) \leq 2H$. To do this, given any $\varepsilon > 0$ we will construct a piecewise smooth compact domain B of X such that the ratio of the area of the boundary ∂B to the volume of B is bounded from above by $2H + \varepsilon$. The inequality $\text{Ch}(X) \leq 2H + \varepsilon$ will then follow from the definition of $\text{Ch}(X)$. As $\varepsilon > 0$ is arbitrary in this construction, we will deduce the desired estimate, thereby proving Assertion 5.11. The construction of the domain B is motivated by a similar construction for estimating the Cheeger constant of certain metric Lie groups in the proof of the main theorem of Peyerimhoff and Samiou in [9]; also see Section 3.8 in [5] for these similar constructions.

For each $n \in \mathbb{N}$ and $t_0 \geq 1$, consider the following domain:

$$B(n, t_0) = \left(\bigcup_{t \in [0, t_0]} l_{h(t)}(\Sigma) \right) \cap \left(\bigcup_{u \in \mathbb{R}} \phi_u(F(n)) \right).$$

The boundary of the box-shaped solid region $B(n, t_0)$ consists of the 'bottom square' face $F(n)$, the 'top square' face $F^{\text{top}}(n, t_0) = [l_{h(t_0)}(\Sigma)] \cap [\bigcup_{u \in \mathbb{R}} \phi_u(F(n))]$ and the 'sides of the box', which is the piecewise smooth surface $S(n, t_0) = \partial B(n, t_0) - [F(n) \cup F^{\text{top}}(n, t_0)]$. By item (2) of Assertion 5.9, $S(n, t_0)$ is contained in the piecewise smooth surface

$$\tilde{S}(n, t_0) = \bigcup_{0 \leq u \leq C_2 t_0} \phi_u(\partial F(n)),$$

see Figure 2.

We next obtain an estimate for the area of the 'top' face $F^{\text{top}}(n, t_0)$. Since Σ satisfies the hypotheses of Lemma 5.2-(2), then Lemma 5.5-(2) ensures that the area function (with respect to g_W) of the related quotient tori $t \mapsto l_{h(t)}(\Sigma)/\Delta$ is exponentially decreasing as $t \rightarrow +\infty$. Therefore, by (5.15), there is some $C_5 > 0$ such that

$$\text{Area}(F^{\text{top}}(n, t_0)) < C_5 n^2 e^{-t_0}. \quad (5.16)$$

We will also need an upper estimate for the area of the 'sides' of $B(n, t_0)$. To do this, consider the function $\tilde{h}: \tilde{S}(n, t_0) \rightarrow [0, C_2 t_0]$ given by $\tilde{h}(\phi_u(q)) = u$, for each $q \in \partial F(n)$. As the gradient $\nabla \tilde{h}$ is clearly orthogonal to $\phi_u(\partial F(n))$, then

$$|\nabla \tilde{h}|(\gamma_q(u)) = g(\nabla \tilde{h}, \gamma'_q(u)) = [\gamma'_q(u)](\tilde{h}) = \frac{d}{du}(\tilde{h} \circ \gamma_q) = 1,$$

which by the coarea formula, gives that

$$\begin{aligned} \text{Area}(S(n, t_0)) &\leq \text{Area}(\tilde{S}(n, t_0)) = \int_0^{C_2 t_0} \left(\int_{\phi_u(\partial F(n))} ds_u \right) du \\ &= \int_0^{C_2 t_0} \text{length}(\phi_u(\partial F(n))) du \stackrel{(a)}{\leq} \int_0^{C_2 t_0} C_1 e^u \text{length}(\partial F(n)) du \end{aligned}$$

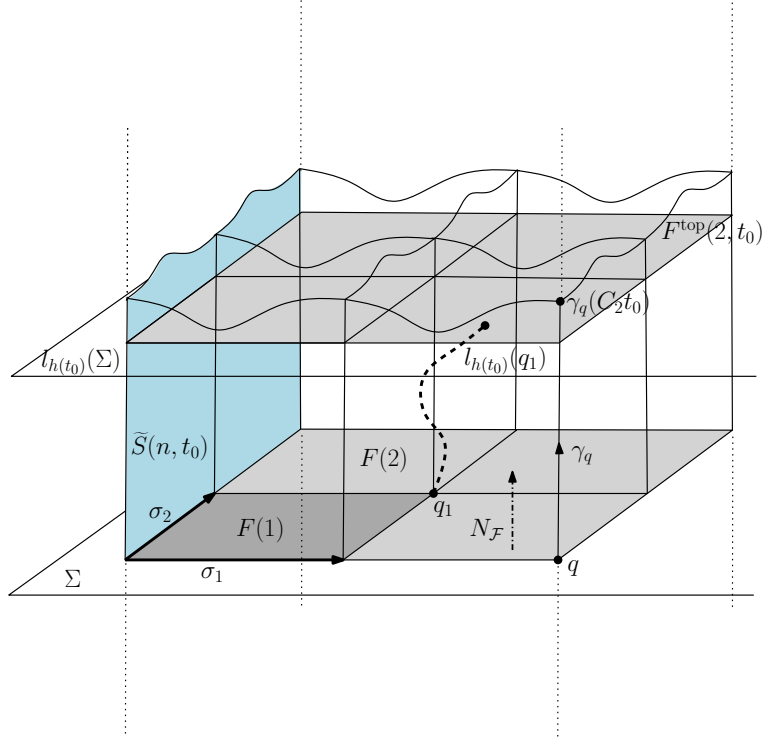


Figure 2: Schematic representation of a 'box'-shaped domain $B(n, t_0)$ for $n = 2$, obtained by intersecting the 'slab' $\bigcup_{t \in [0, t_0]} l_{h(t)}(\Sigma) \subset X$ bounded by Σ and its left translate $l_{h(t_0)}(\Sigma)$, with the vertical 'square column' $\bigcup_{u \in \mathbb{R}} \phi_u(F(n))$ produced by moving the square $F(n) \subset \Sigma$ under the flow $\{\phi_u\}_u$ of the unit normal vector field $N_{\mathcal{F}}$. The bottom and top faces $F(n)$ and $F^{\text{top}}(n, t_0)$ of $B(n, t_0)$ are represented in grey. The side faces $S(n, t_0)$ of $B(n, t_0)$ are contained in larger side faces $\tilde{S}(n, t_0)$ (in blue) obtained by flowing $\partial F(n)$ by ϕ_u , $0 \leq u \leq C_2 t_0$. The dashed curve represents the trajectory of a given point $q \in \Sigma$ under left multiplication by elements in Γ^H , until reaching $l_{h(t_0)}(\Sigma)$; this trajectory might differ from the integral curve γ_q of $N_{\mathcal{F}}$.

$$\stackrel{(5.15)}{\leq} C_1 C_3 n \int_0^{C_2 t_0} e^u du \leq C_1 e^{C_2 t_0} \cdot C_3 n, \quad (5.17)$$

where ds_u denotes the length element in $\phi_u(\partial F(n))$ and in (a) we have used the upper bound for $|(\phi_u)_*(v_p)|$ in item (1) of Assertion 5.9.

We finally prove that for any $\varepsilon > 0$, there exists a $T_0 \geq 1$, such that for any $t_0 \geq T_0$, and n sufficiently large, the next inequality holds:

$$\frac{\text{Area}(\partial B(n, t_0))}{\text{Vol}(B(n, t_0))} \leq 2H + \varepsilon. \quad (5.18)$$

To see that this property holds, fix some $t_0 \geq 1$ and apply the Divergence Theorem to the vector field $N_{\mathcal{F}}$ in the domain $B(n, t_0)$:

$$\begin{aligned} -2H \text{Vol}(B(n, t_0)) &= \int_{B(n, t_0)} \text{div}_X(N_{\mathcal{F}}) = \int_{\partial B(n, t_0)} g(N_{\mathcal{F}}, \eta) \\ &= \int_{F(n)} g(N_{\mathcal{F}}, \eta) + \int_{F^{\text{top}}(n, t_0)} g(N_{\mathcal{F}}, \eta) + \int_{S(n, t_0)} g(N_{\mathcal{F}}, \eta), \end{aligned}$$

where η is the outward pointing unit normal to $B(n, t_0)$ along its boundary. Observe that $\eta = -N_{\mathcal{F}}$ along $F(n)$ and $\eta = N_{\mathcal{F}}$ along $F^{\text{top}}(n, t_0)$. As $g(N_{\mathcal{F}}, \eta) \leq 1$, then we obtain

$$\begin{aligned} -2H \text{Vol}(B(n, t_0)) &\leq -\text{Area}(F(n)) + \text{Area}(F^{\text{top}}(n, t_0)) + \text{Area}(S(n, t_0)) \quad (5.19) \\ &= -\text{Area}(\partial B(n, t_0)) + 2\text{Area}(F^{\text{top}}(n, t_0)) + 2\text{Area}(S(n, t_0)), \end{aligned}$$

which we rewrite as

$$\frac{\text{Area}(\partial B(n, t_0))}{\text{Vol}(B(n, t_0))} \leq 2H + \frac{2\text{Area}(F^{\text{top}}(n, t_0))}{\text{Vol}(B(n, t_0))} + \frac{2\text{Area}(S(n, t_0))}{\text{Vol}(B(n, t_0))}. \quad (5.20)$$

Now,

$$\frac{2\text{Area}(F^{\text{top}}(n, t_0))}{\text{Vol}(B(n, t_0))} = \frac{2\text{Area}(F^{\text{top}}(n, t_0))}{n^2 \text{Vol}(B(1, t_0))} \stackrel{(5.16)}{<} \frac{2C_5 e^{-t_0}}{\text{Vol}(B(1, t_0))} \leq \frac{2C_5 e^{-t_0}}{\text{Vol}(B(1, 1))},$$

which can be made less than $\varepsilon/2$ by taking $t_0 \geq 1$ large enough. For this value of t_0 fixed, we have

$$\frac{2\text{Area}(S(n, t_0))}{\text{Vol}(B(n, t_0))} \stackrel{(5.17)}{\leq} \frac{2 \cdot C_1 e^{C_2 t_0} \cdot C_3 n}{n^2 \text{Vol}(B(1, t_0))}, \quad (5.21)$$

and so the left hand side of (5.21) can be also made less than $\varepsilon/2$ by taking n sufficiently large. With these two estimates, (5.20) implies (5.18), and the proof of Assertion 5.11 is complete. \square

To finish the proof of Lemma 5.7, it remains to prove that if Σ satisfies the hypotheses of Lemma 5.7, then item (E) in the statement of Proposition 5.6 holds. As the volume of the end $\mathcal{D}(0) = \bigcup_{t=0}^{\infty} [l_{h(t)}(\Sigma)/\Delta]$ of W is finite by Lemma 5.5-(2) and since for every $T \in \mathbb{R}$ the mean curvature vector of $\partial \mathcal{D}(T)$ points into $\mathcal{D}(T)$ (Assertion 5.8), we only need to prove the next assertion.

Assertion 5.12. *Let $\Sigma \subset X$ be a surface satisfying the hypotheses of Lemma 5.7. Given $T \in \mathbb{R}$, the domain $\mathcal{D}(T)$ defined in (5.9) is the unique solution to the isoperimetric problem in (W, g_W) for its (finite) value of the enclosed volume.*

Proof. Given $T \in \mathbb{R}$, let $V(T)$ be the volume of $\mathcal{D}(T)$ in (W, g_W) . By Assertion 5.11, the unit normal field $N_{\mathcal{F}/\Delta}$ to the foliation \mathcal{F}/Δ (with respect to the metric g_W) has divergence equal to the negative of twice the mean curvature $H = H(X)$ of the leaves of \mathcal{F}/Δ . By Assertion 5.8, $N_{\mathcal{F}/\Delta}$ restricted to $\partial\mathcal{D}(T)$ points into $\mathcal{D}(T)$. Consider a smooth, possibly non-compact domain $\Omega \subset W$ with volume $V(T)$, $\Omega \neq \mathcal{D}(T)$. Thus, there is a small disk D in the boundary $\partial\Omega$ and an $\varepsilon > 0$ such that

$$-\int_D g_W(N_{\mathcal{F}/\Delta}, N_{\partial\Omega}) \leq \text{Area}(D, g_W) - \varepsilon, \quad (5.22)$$

where $N_{\partial\Omega}$ is the outward pointing unit normal vector field to Ω along its boundary.

Consider the smooth function $f: W \rightarrow \mathbb{R}$ given by

$$f(x) = t \quad \text{provided that } x \in l_{h(t)}(\Sigma)/\Delta, \quad (5.23)$$

(compare with (5.10)). By (5.12) and (5.13), the gradient of f in W satisfies

$$|\nabla f| = |\nabla f| N_{\mathcal{F}/\Delta}, \quad C \leq |\nabla f| \leq 1/C \quad (5.24)$$

for some $C \in (0, 1)$. Since Ω has finite volume, the coarea formula can be applied to f on Ω and gives

$$\text{Vol}(\Omega, g_W) = \int_{-\infty}^{\infty} \left(\int_{\Omega \cap [l_{h(t)}(\Sigma)/\Delta]} \frac{1}{|\nabla f|} dA_t \right) dt \geq C \int_{-\infty}^{\infty} \text{Area}(\Omega \cap [l_{h(t)}(\Sigma)/\Delta]) dt,$$

where dA_t stands for the area element of $\Omega \cap [l_{h(t)}(\Sigma)/\Delta]$ with the induced metric by g_W . Since Ω has finite volume, the last displayed formula implies that there exists a sequence $\{T_n\}_n \subset \mathbb{R}^+$ with $T_n \nearrow \infty$ and a smooth compact increasing exhaustion $W_n = \mathcal{D}(-T_n) - \text{Int}(\mathcal{D}(T_n))$ of W such that for $\Omega_n = W_n \cap \Omega$ and for all n , $\text{Area}(\partial\Omega_n \cap \partial W_n, g_W) \leq \frac{1}{n}$ and $D \subset W_1$.

Applying the Divergence Theorem to $N_{\mathcal{F}/\Delta}$ on Ω_n and letting $N_{\partial\Omega_n}$ be the outward pointing unit normal vector field to Ω_n along its piecewise smooth boundary (note that $N_{\partial\Omega_n} = N_{\partial\Omega}$ on $\partial\Omega \cap \text{Int}(W_n)$), we obtain

$$\begin{aligned} 2 H(X) \text{Vol}(\Omega_n, g_W) &= - \int_{\partial\Omega_n} g_W(N_{\mathcal{F}/\Delta}, N_{\partial\Omega_n}) \\ &= - \int_D g_W(N_{\mathcal{F}/\Delta}, N_{\partial\Omega}) - \int_{\partial\Omega_n - D} g_W(N_{\mathcal{F}/\Delta}, N_{\partial\Omega_n}) \\ &\stackrel{(5.22)}{\leq} \text{Area}(D, g_W) - \varepsilon - \int_{\partial\Omega_n - D} g_W(N_{\mathcal{F}/\Delta}, N_{\partial\Omega_n}) \\ &\leq \text{Area}(\partial\Omega_n, g_W) - \varepsilon, \end{aligned} \quad (5.25)$$

Taking limits as $n \rightarrow \infty$ and using that $\text{Area}(\partial\Omega_n \cap \partial W_n, g_W) \rightarrow 0$, we have $\text{Area}(\partial\Omega_n, g_W) \rightarrow \text{Area}(\partial\Omega, g_W)$ and thus,

$$2 H(X) V(T) = 2 H(X) \text{Vol}(\Omega, g_W) \leq \text{Area}(\partial\Omega, g_W) - \varepsilon. \quad (5.26)$$

Using a similar argument for the domain $\mathcal{D}(T)$, we get

$$2 H(X) V(T) = \text{Area}(l_{h(T)}(\Sigma)/\Delta, g_W) = \text{Area}(\partial\mathcal{D}(T)). \quad (5.27)$$

Equations (5.26) and (5.27) now complete the proof of Assertion 5.12. \square

Finally, Assertions 5.11 and 5.12 imply that Lemma 5.7 holds. \square

Recall that our main goal in this section is to prove Proposition 5.6. To do this, we consider the set

$$\mathcal{A} = \{g \text{ left invariant metric on } \widetilde{\text{SL}}(2, \mathbb{R}) : \text{Proposition 5.6 holds for } g\}.$$

By using the identification in Proposition 4.2, we can view \mathcal{A} as a subset of $\mathcal{M} = (\mathbb{R}^+)^3$. We will prove Proposition 5.6 by showing that $\mathcal{A} = \mathcal{M}$. To do this, we only need to prove that \mathcal{A} is non-empty, open and closed in the connected set $\mathcal{M} \subset \mathbb{R}^3$, which will be the purpose of the next three subsections. By Lemma 5.7, \mathcal{A} can be identified with the set of triples $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{M}$ for which there exists a properly embedded surface Σ satisfying the properties (A), (B) and (C) of Proposition 5.6 on $(\widetilde{\text{SL}}(2, \mathbb{R}), g)$, where g is the left invariant metric associated to $(\lambda_1, \lambda_2, \lambda_3)$.

5.1 \mathcal{A} is non-empty.

Lemma 5.13. *Let X be $\widetilde{\text{SL}}(2, \mathbb{R})$ endowed with a left invariant metric corresponding to the list $(1, 1, 1) \in \mathcal{M}$, see Proposition 4.2. After scaling the metric, $\Pi: X \rightarrow \mathbb{H}^2$ is a Riemannian submersion and the horocylinder $\Sigma_0 = \Pi^{-1}(\alpha_0)$ that appears in Lemma 5.3 satisfies properties (A), (B) and (C) of Proposition 5.6. In particular, by Lemma 5.7 we deduce that $(1, 1, 1) \in \mathcal{A}$.*

Proof. This follows immediately from Lemmas 4.3, 5.3 and 5.7. \square

5.2 Closedness of \mathcal{A} .

We start with a simple consequence of equation (5.8), which we will need in the proof of the closedness of \mathcal{A} .

Lemma 5.14. *Let A be the matrix given in equation (5.5).*

1. Given $c, d \in \mathbb{R}$, the right invariant vector field $c \partial_x + d \partial_y$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is uniformly bounded in $\mathbb{R}^2 \rtimes_A [0, \infty)$.
2. Consider the map $\pi_3: \mathbb{R}^2 \rtimes_A \mathbb{R} \rightarrow \mathbb{R}^2 \rtimes_A \{0\}$ given by $\pi_3(x, y, z) = (x, y, 0)$. Then, there exists $C > 0$ such that for any point $p = (x, y, z) \in \mathbb{R}^2 \rtimes_A [0, \infty)$ and any vector $v_p \in T_p(\mathbb{R}^2 \rtimes_A \{z\})$, the inequality $\|(\pi_3)_*(v_p)\| > C\|v_p\|$ holds.

Now fix the left invariant metric g_0 corresponding to $(1, 1, 1) \in \mathcal{M}$. We will use the properties of $X_0 = (\widetilde{\text{SL}}(2, \mathbb{R}), g_0)$ explained in the proofs of Lemma 5.3 and 5.4, as well as those of the foliations $\mathcal{F}_0 = \mathcal{F}(\Sigma_0, \Gamma^H)$ of X_0 by flat horocylinders defined in (5.4) and \mathcal{F}_0/Δ of $W_0 = (W, (g_0)_W) = (\widetilde{\text{SL}}(2, \mathbb{R})/\Delta, g_0/\Delta)$ by flat tori. Let $N_{\mathcal{F}_0}$, $N_{\mathcal{F}_0/\Delta}$ be the corresponding unit normal vector fields to \mathcal{F}_0 , \mathcal{F}_0/Δ . Recall that in the proof of Lemma 5.4 we identified isometrically X_0 with the space $\mathbb{R}^2 \rtimes_A \mathbb{R}$ appearing in Lemma 5.14, and saw that the unit normal vector field to the foliation \mathcal{F}_0 identifies with ∂_z in $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

Given $g \in \mathcal{A}$, by definition of \mathcal{A} the related space $X = (\widetilde{\text{SL}}(2, \mathbb{R}), g)$ admits a properly embedded surface Σ of constant mean curvature $H(X)$ satisfying properties (A)-...-(E) of Proposition 5.6. Let

$$\mathcal{F} = \mathcal{F}(\Sigma, \Gamma^H) = \{l_h(\Sigma) \mid h \in \Gamma^H\}$$

be the related foliation of X , which induces by Lemma 5.2 a quotient foliation $\mathcal{F}/\Delta = \{l_h(\Sigma)/\Delta \mid h \in \Gamma^H\}$ of (W, g_W) by constant mean curvature tori. In the next lemma, we will consider the foliation \mathcal{F}/Δ as a foliation of the Riemannian manifold W_0 .

Lemma 5.15. *Recall from Lemma 5.5 that the end E_W of (W, g_W) of finite volume has an end representative of the type $\mathcal{D}(0) = \bigcup_{t \in [0, \infty)} [l_{h(t)}(\Sigma_0)/\Delta]$, where $h: \mathbb{R} \rightarrow \Gamma^H$ is the group homomorphism given by (5.3). Given $g \in \mathcal{A}$, the following properties hold:*

1. Each integral curve of $N_{\mathcal{F}_0/\Delta}$ intersects any leaf L' of \mathcal{F}/Δ transversely in a single point and L' can be considered to be the $(g_0)_W$ -normal graph of a smooth, real-valued function defined on Σ_0/Δ . In particular, the function $x \in L' \mapsto \angle_{L'}(x) = (g_0)_x(N_{L'}^0(x), N_{\mathcal{F}_0/\Delta}(x))$, has a constant sign on L' , say positive, where $N_{L'}^0$ is a unit normal vector to L' with respect to the metric g_0 .
2. Given a leaf L' of \mathcal{F}/Δ , let $\varepsilon_{L'} > 0$ be the minimum in L' of the function $\angle_{L'}$ (which exists since $\angle_{L'}$ is continuous on the compact surface L'). Then, $\varepsilon_{L'} = \varepsilon_{L''}$ for every leaf L'' of \mathcal{F}/Δ .
3. Let L be the unique torus leaf of \mathcal{F}/Δ contained in $\mathcal{D}(0)$ which has non-empty intersection with $\partial\mathcal{D}(0) = \Sigma_0/\Delta$, and let $\xi: \Sigma_0/\Delta \rightarrow [0, \infty)$ be the smooth function

that expresses the leaf L as a $(g_0)_W$ -normal graph over Σ_0/Δ given by item (1) of this lemma. If we define

$$a = \max_{x \in \Sigma_0/\Delta} \|\nabla_0 \xi\|_0$$

where the subindex \bullet_0 means that the corresponding object is computed with respect to $(g_0)_W$, then for all $x \in \Sigma_0/\Delta$ we have

$$0 \leq \xi(x) \leq a \cdot \text{diameter}(\Sigma_0/\Delta, (g_0)_W). \quad (5.28)$$

Proof. As $g \in \mathcal{A}$, item (B) of Proposition 5.6 gives that each integral curve of K^H intersects each leaf of \mathcal{F} transversely in a single point. Furthermore, K^P is everywhere tangent to the leaves of \mathcal{F} , by item (2) of Lemma 5.2. These two properties together with equation (5.6) imply that $N_{\mathcal{F}_0}$ is nowhere tangent to any leaf of \mathcal{F} , from where one deduces that each leaf of \mathcal{F} can be locally represented as the (g_0) -normal graph of a smooth function defined on an open set of Σ_0 . As the leaves of \mathcal{F} are topological planes and they have the same double periodicity as Σ_0 , then this local graphing property is actually global. This proves item (1) of the lemma.

As for item (2), since $N_{\mathcal{F}_0}$ is invariant under the left actions of Γ^H and Δ , and the same holds for the foliation \mathcal{F} , a simple compactness argument shows that the minimum angle in W_0 between each leaf of \mathcal{F}/Δ and integral curves of $N_{\mathcal{F}_0/\Delta}$ is the same and bounded from below by some positive number, which gives item (2).

Finally, item (3) is a simple integration argument: given $x_1, x_2 \in \Sigma_0/\Delta$,

$$\xi(x_2) - \xi(x_1) = \int_{\gamma} \|\nabla_0 \xi\|_0 \leq a \cdot \text{length}(\gamma, (g_0)_W)$$

for every piecewise smooth curve $\gamma \subset \Sigma_0/\Delta$ joining x_1, x_2 , from where (5.28) follows directly. Now the proof is complete. \square

Let $\{g_n\}_n \subset \mathcal{A}$ be a sequence of left invariant metrics which converges to a metric $g_\infty \in \mathcal{M}$. Our goal is to prove that $g_\infty \in \mathcal{A}$. Let $X_n = (\widetilde{\text{SL}}(2, \mathbb{R}), g_n)$ and $X_\infty = (\widetilde{\text{SL}}(2, \mathbb{R}), g_\infty)$ be the corresponding homogeneous Riemannian manifolds. As $g_n \in \mathcal{A}$, we have related foliations \mathcal{F}_n of $W_n = (W, (g_n)_W)$ with leaves of constant mean curvature $H(X_n)$. By item (2) of Lemma 5.15, for each $n \in \mathbb{N}$ the angle functions in W_0 between integral curves of $N_{\mathcal{F}_0/\Delta}$ and leaves of \mathcal{F}_n are bounded from below by some $\varepsilon_n > 0$. By item (3) of the same Lemma 5.15, for every $n \in \mathbb{N}$ we have a unique torus leaf L_n of \mathcal{F}_n/Δ contained in $\mathcal{D}(0)$ such that $L_n \cap \partial\mathcal{D}(0) \neq \emptyset$, and L_n can be expressed as the $(g_0)_W$ -normal graph of a smooth function $\xi_n: \Sigma_0/\Delta \rightarrow [0, \infty)$. We next analyze when one can take limits on these objects.

Lemma 5.16. *In the above situation, suppose that the sequence $\{\varepsilon_n\}_n$ is bounded away from zero. Then, a subsequence of the graphing functions $\xi_n: \Sigma_0/\Delta \rightarrow [0, \infty)$ converges*

smoothly on Σ_0/Δ to a smooth function $\xi_\infty: \Sigma_0/\Delta \rightarrow [0, \infty)$, and the $(g_0)_W$ -normal of ξ_∞ over Σ_0/Δ defines a surface L_∞ of constant mean curvature $H(X_\infty)$ in $(W, (g_\infty)_W)$, whose lifting \tilde{L} to $\widetilde{\text{SL}}(2, \mathbb{R})$ of L through the projection π_W defined in (5.1) produces a properly embedded surface that satisfies the hypotheses of Lemma 5.7.

Proof. Lemma 5.14 and the fact that $N_{\mathcal{F}_0}$ identifies with ∂_z (see the first paragraph after the statement of Lemma 5.14) imply that giving an upper bound for $\|\nabla_0 \xi_n\|_0$ is equivalent to giving a bound by below for the corresponding angle function \angle_{L_n} defined in item (1) of Lemma 5.15. Moreover, if the angle functions \angle_{L_n} are uniformly bounded away from zero, then (5.8) implies that $\{\|\nabla_0 \xi_n\|_0\}_n$ is uniformly bounded from above in Σ_0/Δ . As (5.28) gives a uniform bound for $\{\xi_n\}_n$, then a standard argument based on the Arzelá-Ascoli theorem and regularity results in elliptic theory produces a subsequence of $\{\xi_n\}_n$ that converges smoothly on Σ_0/Δ to a smooth function $\xi_\infty: \Sigma_0/\Delta \rightarrow [0, \infty)$ whose $(g_0)_W$ -normal graph over Σ_0/Δ defines a surface L_∞ of constant mean curvature $H = \lim_n H(X_n)$ in $(W, (g_\infty)_W)$. Finally, the lifting \tilde{L} to $\widetilde{\text{SL}}(2, \mathbb{R})$ of L through π_W is a properly embedded surface that satisfies the hypotheses of Lemma 5.7 (properties (A), (C) of Proposition 5.6 are preserved under smooth limits, and property (B) of Proposition 5.6 holds by construction). Therefore, Lemma 5.7 implies that $H = H(X_\infty) = \frac{1}{2}\text{Ch}(X_\infty)$ and the proof is complete. \square

Lemma 5.17. *\mathcal{A} is a closed subset of \mathcal{M} .*

Proof. Let $\{g_n\}_n \subset \mathcal{A}$ be a sequence of left invariant metrics which converges to a metric $g_\infty \in \mathcal{M}$. During this proof, we will use the notation stated in the paragraph before Lemma 5.16. By Lemmas 5.7 and 5.16, it suffices to show that the sequence $\{\varepsilon_n\}_n$ is bounded away from zero. Arguing by contradiction, assume that after extracting a subsequence, one has $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

We first make three observations:

- (O1) $\text{Ch}(X_\infty) = \lim_{n \rightarrow \infty} \text{Ch}(X_n)$: this follows from the definition of the Cheeger constant and from the fact that the metrics g_n converge uniformly to g .
- (O2) For each $n \in \mathbb{N}$ we have $\text{Ch}(X_n) = 2H(X_n)$: This is a consequence of Lemma 5.7, as $g_n \in \mathcal{A}$.
- (O3) $\text{Ch}(X_\infty) > 0$, as every left invariant metric on $\widetilde{\text{SL}}(2, \mathbb{R})$ has this property.

By the invariance of the foliation \mathcal{F}_n under the left action of Γ^H , we can assume that for each $n \in \mathbb{N}$ there exists a point $p_n \in \Sigma_0/\Delta$ such that the leaf $L(p_n)$ of \mathcal{F}_n/Δ passing through p_n makes an angle (with respect to g_0) of $\varepsilon_n > 0$ with $N_{\mathcal{F}_0/\Delta}$ at the point p_n . Since the metrics g_n converge uniformly to g , then there are uniform estimates for the norms of the second forms of all of these leaves $L(p_n) \subset W_n$ by the curvature estimates

in [11]. Therefore, after replacing by a subsequence, the points p_n converge to some point $p_\infty \in \Sigma_0/\Delta$ and there exists a complete, connected, immersed two-sided surface $\tilde{L}_\infty \subset X$ which is invariant under Δ , with $p_\infty \in L_\infty := \tilde{L}_\infty/\Delta$, with constant mean curvature $\frac{1}{2}\text{Ch}(X_\infty)$ in $(\widetilde{\text{SL}}(2, \mathbb{R}), g_\infty)$, and such that L_∞ is a smooth limit of portions of the leaves $L(p_n)$. In particular, L_∞ is stable and the same holds for its lifting \tilde{L}_∞ to X . Since L_∞ is a smooth limit of portions of the $L(p_n)$, then K^P is everywhere tangent to L_∞ . As the angle with respect to $(g_0)_W$ of $L(p_n)$ with $N_{\mathcal{F}_0/\Delta}$ goes to zero as $n \rightarrow \infty$, then $N_{\mathcal{F}_0/\Delta}$ is tangent to L_∞ at p_∞ . Note that the inner product (with respect to $(g_\infty)_W$) of $N_{\mathcal{F}_0/\Delta}$ with the unit normal vector field N_{L_∞} to L_∞ cannot change sign on L_∞ , as the same property holds if we exchange L_∞ by $L(p_n)$ and $(g_\infty)_W$ by $(g_n)_W$ for all n , by item (1) of Lemma 5.15. Hence we can assume $(g_\infty)_W(N_{\mathcal{F}_0/\Delta}, N_{L_\infty}) \geq 0$ on L_∞ . Using Lemma 5.4 and the fact that K^P is everywhere tangent to L_∞ , we deduce that

$$(g_\infty)_W(N_{\mathcal{F}_0/\Delta}, N_{L_\infty}) = (g_\infty)_W(K^H, N_{L_\infty}). \quad (5.29)$$

As K^H is a Killing field on X_∞ , then the function $u = (g_\infty)_W(K^H, N_{L_\infty})$ satisfies the Jacobi equation on L_∞ . Equation (5.29) implies that $u \geq 0$ on L_∞ with a zero at p_∞ , hence u is everywhere zero on L_∞ by the maximum principle.

Since the linearly independent right invariant vector fields K^P, K^H on $\widetilde{\text{SL}}(2, \mathbb{R})$ are everywhere tangent to \tilde{L}_∞ , then \tilde{L}_∞ is the left coset of some two-dimensional subgroup of $\widetilde{\text{SL}}(2, \mathbb{R})$. In particular, \tilde{L}_∞ is ambiently isometric to a two-dimensional subgroup of $\widetilde{\text{SL}}(2, \mathbb{R})$. This is a contradiction, as Corollary 3.17 in [5] ensures that two-dimensional subgroups of $\widetilde{\text{SL}}(2, \mathbb{R})$ with a left invariant metric have zero mean curvature, and the mean curvature of \tilde{L}_∞ is $\frac{1}{2}\text{Ch}(X_\infty) > 0$ by Observation (O3). This contradiction completes the proof of the lemma. \square

5.3 Openness of \mathcal{A} .

Lemma 5.18. *\mathcal{A} is open in \mathcal{M} .*

Proof. Fix $g \in \mathcal{A}$. We will show that metrics in \mathcal{M} that are sufficiently close to g are also in \mathcal{A} . Let $X = (\widetilde{\text{SL}}(2, \mathbb{R}), g)$ and for g' sufficiently close to g , let $X' = (\widetilde{\text{SL}}(2, \mathbb{R}), g')$. By Lemma 5.7, it suffices to prove that there is a properly embedded surface $\Sigma' \subset X'$ of constant mean curvature satisfying items (A), (B), (C) of Proposition 5.6 for g' sufficiently close to g . As $g \in \mathcal{A}$, there exists a properly embedded surface $\Sigma \subset X$ with constant mean curvature $H(X)$ satisfying Proposition 5.6. Consider the quotient torus $T = \Sigma/\Delta$ of constant mean curvature $H(X)$ in (W, g_W) .

Note that the existence of the product foliation \mathcal{F}/Δ of (W, g_W) by tori of constant mean curvature $H(X)$ given by item (D) of Proposition 5.6, implies that T admits a positive Jacobi function J . Namely, one can choose $J = g_W(K^H, N_T)$ for a unit normal

vector field N_T to T ; observe that equation (5.6) implies that although the vector field K^H does not descend to W , the inner product with respect to the metric g of K^H with the unit normal vector to Σ descends to the quotient torus T as K^P is everywhere tangent to Σ . This implies that the space of Jacobi functions on T is generated by J , and $\int_T J \neq 0$. By Proposition 7.4 in the Appendix, for $g' \in \mathcal{M}$ sufficiently close to g , there is an embedded constant mean curvature torus T' in (W, g'_W) that is smoothly close to T , and the space of Jacobi functions on T' is one-dimensional, generated by a smooth function $J': T' \rightarrow \mathbb{R}$ with $\int_{T'} J' \neq 0$.

Since every integral curve of K^H intersects transversely Σ at a single point and both $T, T' \subset W$ are compact and arbitrarily close, then every integral curve of K^H intersects transversely $\Sigma' := \pi_W^{-1}(T') \subset \widetilde{\text{SL}}(2, \mathbb{R})$ at a single point, where π_W is the projection defined in (5.2). In other words, Σ' satisfies property (B) of Proposition 5.6. Property (C) of the same proposition holds for Σ' by construction.

The next argument shows that Σ' satisfies property (A) of Proposition 5.6: otherwise the inner product with respect to g' of K^P with the unit normal vector field to Σ' defines a non-zero Jacobi function on Σ' . As K^P is invariant under the left action of Δ (Lemma 5.2), then the function $g'_W(K^P, N_{T'})$ is well-defined on T' ($N_{T'}$ stands for the unit normal vector field to T') thereby producing a non-zero Jacobi function on T' . As the space of Jacobi functions on T' is one-dimensional, then $g'_W(K^P, N_{T'})$ is a non-zero multiple of the function J' , and thus, $\int_{T'} g'_W(K^P, N_{T'}) \neq 0$. This is impossible, since the divergence of K^P in (W, g'_W) is zero and we contradict the Divergence Theorem applied to K^P on the end of finite volume bounded by T' in (W, g'_W) .

Therefore, for g' sufficiently close to g , the lifted surface $\Sigma' \subset X'$ satisfies conditions (A), (B) and (C) of Proposition 5.6. By Lemma 5.7, Σ' satisfies the remaining properties (D) and (E) of Proposition 5.6, that is, $g' \in \mathcal{A}$ and the proof of the lemma is complete. \square

5.4 Proof of Proposition 5.6.

By Lemmas 5.13, 5.17 and 5.18, $\mathcal{A} \subset \mathcal{M}$ is a non-empty subset of \mathcal{M} that is both open and closed. Since \mathcal{M} is connected, then $\mathcal{A} = \mathcal{M}$, which by definition of \mathcal{A} as the subset of the metrics \mathcal{M} of $\widetilde{\text{SL}}(2, \mathbb{R})$ for which the Proposition 5.6 holds, completes the proof of the proposition. \square

Remark 5.19. Let X be isometric to $\widetilde{\text{SL}}(2, \mathbb{R})$ equipped with a left invariant metric and let \mathcal{F} be the foliation described in Proposition 5.6. Recall that \mathcal{F} is invariant under left translations by elements in $\Gamma^H \cup \Gamma^P$. In particular, \mathcal{F} is invariant under left translations by elements in the two-dimensional subgroup \mathbb{H}_θ of $\widetilde{\text{SL}}(2, \mathbb{R})$ generated by $\Gamma^H \cup \Gamma^P$. Since every element $a \in \widetilde{\text{SL}}(2, \mathbb{R})$ can be expressed uniquely as $a = bc$ where $b \in \Gamma^E$ and $c \in \mathbb{H}_\theta$, then every left translation of \mathcal{F} can be expressed as $b\mathcal{F}$ for some $b \in \Gamma^E$. As

the center Z of $\widetilde{\text{SL}}(2, \mathbb{R})$ is an infinite cyclic subgroup contained in Γ^E and elements in Z leave invariant \mathcal{F} , then the collection of left translations of \mathcal{F} can be parameterized by the \mathbb{S}^1 -family Γ^E/Z .

6 The proof of Theorems 1.4 and 1.5

Let X be a non-compact, simply connected homogeneous three-manifold. First suppose that $\text{Ch}(X) = 0$. In this setting, item (1) of Theorem 1.4 follows from Remark 2.1 and from item (2) of Lemma 2.3. Item (2) of Theorem 1.4 is a consequence of Lemma 2.4 and Theorem 3.1. Finally, item (3) of Theorem 1.4 also follows from Theorem 3.1.

Next consider the case $\text{Ch}(X) > 0$. In this situation, X can be isometrically identified with either $\widetilde{\text{SL}}(2, \mathbb{R})$ endowed with a left invariant metric or a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $\text{trace}(A) > 0$, equipped with its canonical metric. We separate the proof of Theorem 1.4 when $\text{Ch}(X) > 0$ into two cases.

Case A: X is $\widetilde{\text{SL}}(2, \mathbb{R})$ with a left invariant metric.

In order to prove several parts of Theorem 1.4 in this Case A, we will use the product foliation \mathcal{F} given in Proposition 5.6 by surfaces of mean curvature $H(X) = \frac{1}{2}\text{Ch}(X)$. Let M be a compact immersed surface in X with constant mean curvature. Then there exists a unique leaf Σ' of \mathcal{F} such that M lies on the mean convex side of Σ' and intersects Σ' at some point p . Since M is compact and Σ' is non-compact and properly embedded in X , these surfaces are different, and an application of the maximum principle implies that the absolute mean curvature function of M at the point p is greater than $\frac{1}{2}\text{Ch}(X)$. The inequality $\frac{1}{2}\text{Ch}(X) < H$ where $H > 0$ is the constant mean curvature of the boundary of any isoperimetric domain in X (in item (1) of Theorem 1.4) then follows. The inequality $\text{Ch}(X) < \frac{I(t)}{t}$ for all $t > 0$ follows from item (2) of Lemma 2.3, and then Remark 2.1 finishes the proof of item (1) of Theorem 1.4. Item (2) of Theorem 1.4 follows from Lemma 2.4 and Assertion 5.11. Hence it only remains to prove item (3) of Theorem 1.4 in this Case A.

In the sequel, \mathcal{F} will denote the foliation of X that appears in Proposition 5.6.

Lemma 6.1. *There exist positive constants C, τ such that if Ω is an isoperimetric domain in X with volume greater than 1, then:*

1. *The norms of the second fundamental forms of $\partial\Omega$ and of the leaves of \mathcal{F} are bounded from above by C .*
2. *The injectivity radius of $\partial\Omega$ and of the leaves of \mathcal{F} are both greater than 4τ .*
3. *For any $p \in \partial\Omega$ and $t \in (0, 2\tau]$, we have*

$$\frac{1}{2}\pi t^2 \leq \text{Area}(B_{\partial\Omega}(p, t)) \leq 2\pi t^2. \quad (6.1)$$

4. If Δ is a maximal collection of pairwise disjoint geodesic disks of radius τ in $\partial\Omega$, then

$$\text{Area}(\Delta) \geq \frac{1}{16} \text{Area}(\partial\Omega). \quad (6.2)$$

Proof. That item (1) of the lemma holds for $\partial\Omega$ was explained at the beginning of Section 2. Furthermore, item (1) also holds for the leaves of the product foliation \mathcal{F} since all these leaves have compact quotients and are isometric to each other by an ambient left translation, see Proposition 5.6.

In addition, note that for any $C > 0$ there exists a $\tau > 0$ such that the following property holds: For any complete surface in X such that the norm of its second fundamental form is bounded by C , the injectivity radius of this surface is greater than 4τ . This proves that item (2) of the lemma holds.

By the Gauss equation and items (1) and (2), the Gaussian curvature of $\partial\Omega$ is uniformly bounded and the inequalities in (6.1) hold for a possibly smaller τ .

Let $\Delta = \{B_1, \dots, B_k\}$ be a maximal collection of pairwise disjoint geodesic disks of radius τ in $\partial\Omega$ and let $\Delta' = \{B'_1, \dots, B'_k\}$ the related sequence of geodesic disks of radius 2τ with B_i having the same center as the corresponding B_i , $i = 1, \dots, k$. By the triangle inequality, as the collection Δ is maximal, then the collection of disks Δ' is a covering of $\partial\Omega$. Then,

$$\text{Area}(\Delta) = \sum_{i=1}^k \text{Area}(B_i) \stackrel{(6.1)}{\geq} k \frac{\pi\tau^2}{2} \stackrel{(6.1)}{\geq} \frac{1}{16} \sum_{i=1}^k \text{Area}(B'_i) \stackrel{(*)}{\geq} \frac{1}{16} \text{Area}(\partial\Omega),$$

where $(*)$ follows from the fact that Δ' is a covering of $\partial\Omega$. \square

Given an $a \in X$, let \mathcal{F}_a denote the foliation obtained by left translating \mathcal{F} by a . As we explained in Remark 5.19, the set $\{\mathcal{F}_a \mid a \in X\}$ is an \mathbb{S}^1 -family of foliations of X , all whose leaves have constant mean curvature $H(X)$. We will use this compactness property for the family of left translations of \mathcal{F} in the proof of the next theorem; items (1) and (2) of Theorem 6.2 will complete then the proof of item (3) of Theorem 1.4.

Theorem 6.2. *Given a sequence $\{\Omega_n\}_n$ of isoperimetric domains in X with volumes tending to infinity, there exist open sets $S_n \subset \partial\Omega_n$ with*

$$\frac{\text{Area}(S_n)}{\text{Area}(\partial\Omega_n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty, \quad (6.3)$$

such that for any sequence of points $q_n \in S_n$, there exists a subsequence of the surfaces $\{q_n^{-1}\partial\Omega_n\}_n$ that converges smoothly (in the uniform topology on compact sets of X) to the leaf Σ_a of some \mathcal{F}_a passing through e . Furthermore, for this subsequence, the domains $q_n^{-1}\Omega_n$ converge to the closure of the mean convex component of $X - \Sigma_a$. In particular:

1. *The radii of the Ω_n tend to infinity.*

2. The mean curvatures of $\partial\Omega_n$ converge to $H(X)$.

Proof. Let $N_{\mathcal{F}}$ denote the unit normal vector field to the foliation \mathcal{F} , which has divergence $-2H(X)$. Let $\Omega \subset X$ be an isoperimetric domain. By the Divergence Theorem,

$$2H(X)\text{Vol}(\Omega) = \int_{\partial\Omega} \langle N_{\mathcal{F}}, N_{\partial\Omega} \rangle \leq \text{Area}(\partial\Omega), \quad (6.4)$$

where $N_{\partial\Omega}$ is the inward pointing unit normal of $\partial\Omega$. By the already proven item (2) of Theorem 1.4, we see that the quantity $\frac{\text{Area}(\partial\Omega)}{\text{Vol}(\Omega)}$ tends to $2H(X)$ as $\text{Vol}(\Omega)$ tends to infinity. Hence, from (6.4) we see that given $\varepsilon \in (0, 1)$, there exists some $V(\varepsilon) > 1$ such that if $\text{Vol}(\Omega) > V(\varepsilon)$, then

$$1 - \varepsilon^4 \leq \frac{\int_{\partial\Omega} \langle N_{\mathcal{F}}, N_{\partial\Omega} \rangle}{\text{Area}(\partial\Omega)} \leq 1. \quad (6.5)$$

Let $I_\varepsilon \subset \partial\Omega$ denote the closed subset of all points $p \in \partial\Omega$ such that

$$\langle N_{\mathcal{F}}, N_{\partial\Omega} \rangle(p) \leq 1 - \varepsilon^2.$$

For a generic choice of ε , I_ε is a smooth compact subdomain of $\partial\Omega$. In fact, since closed sets are measurable, for every $\varepsilon > 0$ the area functional of $\partial\Omega$ makes sense on I_ε and $\text{Area}(I_\varepsilon) + \text{Area}(\partial\Omega - I_\varepsilon) = \text{Area}(\partial\Omega)$. It then follows from (6.5) that

$$1 - \varepsilon^4 \leq \frac{(1 - \varepsilon^2)\text{Area}(I_\varepsilon) + \text{Area}(\partial\Omega - I_\varepsilon)}{\text{Area}(\partial\Omega)} = 1 - \varepsilon^2 \frac{\text{Area}(I_\varepsilon)}{\text{Area}(\partial\Omega)},$$

from where we get

$$\frac{\text{Area}(I_\varepsilon)}{\text{Area}(\partial\Omega)} \leq \varepsilon^2. \quad (6.6)$$

Let Δ be a maximal collection of closed, pairwise disjoint geodesic disks of radius $\tau > 0$ in $\partial\Omega$ (where τ was defined in Lemma 6.1). Given $\varepsilon > 0$, a disk D in Δ is called an ε -good disk if $\text{Area}(D \cap I_\varepsilon) < \varepsilon$, and an ε -bad disk otherwise. We will denote by Δ_B the subcollection of ε -bad disks in Δ .

Next we will prove the following property:

(P) Given $\delta > 0$, there exists $\varepsilon \in (0, \delta)$ such that if $\Omega \subset X$ is an isoperimetric domain with volume greater than $V(\varepsilon)$ (given so that (6.5) holds), then for any maximal collection of pairwise disjoint geodesic disks of radius τ in $\partial\Omega$, the ratio of the number of ε -bad disks to the total number of disks in Δ is less than δ .

Otherwise, we can find $\delta > 0$ for which given any $\varepsilon \in (0, \delta)$, there exists an isoperimetric domain $\Omega \subset X$ with $\text{Volume}(\Omega) > V(\varepsilon)$ and a maximal collection Δ of pairwise disjoint

geodesic disks of radius τ in $\partial\Omega$ for which the ratio of the number of ε -bad disks in Δ to the total number of disks in Δ is not less than δ . Take any $\varepsilon \in (0, \delta)$ and consider the set I_ε defined above. Then,

$$\text{Area}(I_\varepsilon) \geq \text{Area}(\Delta \cap I_\varepsilon) \geq \sum_{D \in \Delta_B} \text{Area}(D \cap I_\varepsilon) \geq \varepsilon \#(\Delta_B) \geq \varepsilon \delta \#(\Delta),$$

where $\#(A)$ denotes the cardinality of a set A . From (6.1), (6.2) and the fact that the disks in Δ are pairwise disjoint, we deduce that

$$32\pi\tau^2 \#(\Delta) \geq \text{Area}(\partial\Omega).$$

From the last two displayed inequalities and (6.6), we get

$$\frac{\varepsilon\delta}{32\pi\tau^2} \leq \frac{\text{Area}(I_\varepsilon)}{\text{Area}(\partial\Omega)} \leq \varepsilon^2,$$

which is a contradiction if ε is chosen small enough. This proves property **(P)**.

We now prove Theorem 6.2. Consider a sequence $\{\Omega_n\}_n$ of isoperimetric domains in X with $\text{Volume}(\Omega_n) \rightarrow \infty$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, let Δ_n be a maximal collection of pairwise disjoint geodesic disks in $\partial\Omega_n$ of radius τ . Let $\delta_n = \frac{1}{n}$, $n \in \mathbb{N}$. By property **(P)**, there exists $\varepsilon_n \in (0, \frac{1}{n})$ and a subsequence of $\{\Omega_n\}_n$ (denoted in the same way) such that $\text{Volume}(\Omega_n) > V(\varepsilon_n)$ and

$$\frac{\#(\Delta_B(n))}{\#(\Delta_n)} < \frac{1}{n}, \quad (6.7)$$

where $\Delta_B(n)$ is the collection of ε_n -bad disks in Δ_n . We will prove that the conclusions in Theorem 6.2 hold for this subsequence (note that this is enough to conclude that Theorem 6.2 holds for the original sequence $\{\Omega_n\}_n$).

Next we define the open set $S_n := \partial\Omega_n - \mathcal{A}_n$, where

$$\mathcal{A}_n = \{p \in \partial\Omega_n \mid \text{dist}_{\partial\Omega_n}(p, \Delta_B(n)) \leq \tau\}.$$

We now prove that the sequence $\{S_n\}_n$ satisfies (6.3). Write $\Delta_B(n) = \{D_1, \dots, D_k\}$ and let $D'_i \subset \partial\Omega$ be the disk of radius 2τ with the same center as D_i , $i = 1, \dots, k$. Using (6.1) we have $\text{Area}(D'_i) \leq 8\pi\tau^2$ for every $i = 1, \dots, k$. By the triangle inequality, \mathcal{A}_n is contained in $D'_1 \cup \dots \cup D'_k$. Therefore,

$$\text{Area}(\mathcal{A}_n) \leq 8\pi\tau^2 \#(\Delta_B(n)),$$

and

$$\frac{\text{Area}(\mathcal{A}_n)}{\text{Area}(\partial\Omega_n)} \leq \frac{\text{Area}(\mathcal{A}_n)}{\text{Area}(\Delta_n)} \leq 8\pi\tau^2 \frac{\#(\Delta_B(n))}{\text{Area}(\Delta_n)} \stackrel{(6.1)}{\leq} 8\pi\tau^2 \frac{\#(\Delta_B(n))}{\frac{\pi}{2}\tau^2 \#(\Delta_n)} \stackrel{(6.7)}{<} \frac{16}{n},$$

from where (6.3) follows.

Now consider a sequence $q_n \in S_n$, $n \in \mathbb{N}$. Since the set of foliations $\mathcal{F}_{q_n^{-1}}$ lies in the \mathbb{S}^1 -family of foliations $\{\mathcal{F}_a \mid a \in X\}$ (here we are using the notation introduced just before the statement of Theorem 6.2), then after choosing a subsequence, we can assume that the $\mathcal{F}_{q_n^{-1}}$ converge as $n \rightarrow \infty$ to \mathcal{F}_a for some $a \in X$. Observe that the constant values H_n of the mean curvatures of $q_n^{-1}\partial\Omega_n$ lie in some compact interval of $(0, \infty)$, since $H(X) > 0$ by the already proven item (2) of Theorem 1.4. Also, since the norms of the second fundamental forms of the surfaces $q_n^{-1}\partial\Omega_n$ are bounded from above, then Theorem 3.5 in [6] implies that each $q_n^{-1}\partial\Omega_n$ has a regular neighborhood inside $q_n^{-1}\Omega_n$ of radius greater than some $r_0 > 0$, where r_0 only depends on X and on the uniform bound of the second fundamental forms of the surfaces $q_n^{-1}\partial\Omega_n$. A standard compactness argument from elliptic theory (see [7] for this type of argument) proves that a subsequence of the regions $q_n^{-1}\Omega_n$ converges to a properly immersed, three-dimensional domain $\mathcal{D} \subset X$ that is *strongly Alexandrov embedded*, i.e., there exists a complete Riemannian three-manifold W with boundary and a proper isometric immersion $f: W \rightarrow X$ that is injective on the interior of W such that $f(W) = \mathcal{D}$. Furthermore, the boundary $\partial\mathcal{D}$ is a possibly disconnected surface of positive constant mean curvature, which might not be embedded but still satisfies that a small fixed normal variation of $\partial\mathcal{D}$ into \mathcal{D} is an embedded surface. The boundary surface $\partial\mathcal{D}$ also has a regular r_0 -neighborhood in \mathcal{D} .

By construction, one of the components P of $\partial\mathcal{D}$ passes through the origin, and we claim that P equals the leaf Σ_a of \mathcal{F}_a passing through the origin. By the defining property of the set S_n and the fact that the collection Δ_n is maximal, we can deduce that since $q_n \in S_n$, there exists a geodesic τ -disk $D_n \in \Delta_n - \Delta_B(n)$ which is at an intrinsic distance less than τ from q_n . Clearly the disks $q_n^{-1}D_n$ converge as $n \rightarrow \infty$ to a geodesic disk $D_\infty \subset P$ of radius τ . Since D_n is an ε_n -good disk and $\varepsilon_n \rightarrow 0$, then for all points $q \in D_\infty \subset P$ the unit normal to P must be equal to the unit normal of the leaf of the foliation \mathcal{F}_a passing through q . This implies that D_∞ is contained in some leaf of \mathcal{F}_a and by analytic continuation, we deduce that $P = \Sigma_a$.

We claim that $\partial\mathcal{D} = \Sigma_a$. As explained in Remark 5.19, there exists an element $b \in X$ such that after some fixed left translation, we may assume that $b\mathcal{F}_a = \mathcal{F} = \mathcal{F}(\Sigma, \Gamma^H)$ and $b\Sigma_a = \Sigma$ (here we are using the notation in Proposition 5.6). Call $\tilde{\mathcal{D}} = b\mathcal{D}$ and note that Σ is a component of $\partial\tilde{\mathcal{D}}$. To show $\partial\mathcal{D} = \Sigma_a$, it suffices to prove that $\partial\tilde{\mathcal{D}} = \Sigma$.

By item (C) of Proposition 5.6, $\mathcal{F} = \{l_{h(t)}(\Sigma) \mid t \in \mathbb{R}\}$, where $h(t)$ is the parametrization of Γ^H given by (5.3). Consider the distance in X from Σ to $l_{h(t)}(\Sigma)$, as a function of t . This function is continuous because it is the lifting of the corresponding distance function between leaves of the associated quotient product torus foliation of W described in item (D) of Proposition 5.6, which is clearly continuous. The continuity of $t \mapsto \text{dist}_X(\Sigma, l_{h(t)}(\Sigma))$ and the existence of a fixed size regular neighborhood of $\Sigma \subset \partial\tilde{\mathcal{D}}$ in $\tilde{\mathcal{D}}$ (Theorem 3.5 in [6]) implies that for $t_1 > 0$ sufficiently small, each of the leaves

$l_{h(t)}(\Sigma)$ with $t \in (0, t_1]$, is contained in the interior of $\tilde{\mathcal{D}}$. Consider now for each $t \in (0, t_1]$ the related subdomain $\tilde{\mathcal{D}}(t)$ of $\tilde{\mathcal{D}}$ whose boundary is $(\partial\tilde{\mathcal{D}} - \Sigma) \cup l_{h(t)}(\Sigma)$. Observe that $\tilde{\mathcal{D}}(t)$ is also strongly Alexandrov embedded, and by the regular neighborhood theorem, $l_{h(t)}(\Sigma)$ has an r_0 -regular neighborhood in $\tilde{\mathcal{D}}(t)$. In particular, we can continue to consider the deformations $\tilde{\mathcal{D}}(t)$ of $\tilde{\mathcal{D}}$ by increasing the value of t , and so by a continuity argument, define $\tilde{\mathcal{D}}(t)$ for all $t \in (0, \infty)$. Therefore $\partial\tilde{\mathcal{D}} = \Sigma$, and so $\partial\mathcal{D} = \Sigma_a$, as claimed. In particular, \mathcal{D} coincides with the mean convex component of $X - \Sigma_a$. As the domains $q_n^{-1}\Omega_n$ converge to \mathcal{D} and \mathcal{D} contains geodesic balls in X of arbitrarily large radius, we conclude that the radii of the manifolds $q_n^{-1}\Omega_n$ tend to infinity, and the same holds for the radii of the Ω_n . As $\{\Omega_n\}_n$ is a subsequence of an arbitrary sequence of isoperimetric regions with volumes tending to infinity, it follows that the radii of any such sequence also tend to infinity. Finally, the mean curvatures of Ω_n converge to $H(X)$ since $\Sigma_a \subset \partial\mathcal{D}$ has mean curvature $H(X)$. This concludes the proof of Theorem 6.2. \square

Case B: X is a non-unimodular semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric.

The proof of Theorem 1.4 given when $X = \widetilde{\text{SL}}(2, \mathbb{R})$ with a left invariant metric can be easily modified for the remaining case where $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with $\text{trace}(A) > 0$. We next explain the main aspects of this modification and leave the details to the reader.

Let $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ be the foliation by horizontal planes in $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$. By Proposition 2.6, all these planes have constant mean curvature $H(X) = \frac{1}{2}\text{Ch}(X) = \frac{1}{2}\text{trace}(A) > 0$, and are everywhere tangent to the left invariant vector fields E_1, E_2 defined in (4.1). This foliation \mathcal{F} plays the role of the foliation appearing in Proposition 5.6.

Let Σ be the plane $\mathbb{R}^2 \rtimes_A \{0\}$, which is a normal subgroup of X . Consider two linearly independent elements $a_1, a_2 \in \Sigma$ and let Δ be the $(\mathbb{Z} \times \mathbb{Z})$ -subgroup of X generated by the left translations by a_1, a_2 . The role of the parabolic 1-parameter subgroup Γ^P appearing when X was $\widetilde{\text{SL}}(2, \mathbb{R})$ will be now played by the 1-parameter subgroup $\{ta_1 \mid t \in \mathbb{R}\} \subset \Sigma$. With these adaptations, it is now easy to finish the proof of Theorem 1.4 in the non-unimodular case $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$.

Since in this Case B, \mathcal{F} is invariant under left translation by arbitrary elements of X , one can prove the following (stronger) analogue of Theorem 6.2.

Theorem 6.3. *Given a sequence $\{\Omega_n\}_n$ of isoperimetric domains in X with volumes tending to infinity, there exist open sets $S_n \subset \partial\Omega_n$ with*

$$\frac{\text{Area}(S_n)}{\text{Area}(\partial\Omega_n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty, \quad (6.8)$$

such that for any sequence of points $q_n \in S_n$, the surfaces $\{q_n^{-1}\partial\Omega_n\}_n$ converge smoothly (in the uniform topology on compact sets of X) to $\Sigma = \mathbb{R}^2 \rtimes_A \{0\}$.

Furthermore:

1. For this sequence, the domains $q_n^{-1}\Omega_n$ converge to $\mathbb{R}^2 \rtimes_A [0, \infty)$.
2. The radii of the Ω_n tend to infinity.
3. The mean curvatures of $\partial\Omega_n$ converge to $H(X)$.

This concludes the proof of Theorem 1.4.

Corollary 6.4. *Let X be a non-compact, simply connected homogeneous three-manifold. Then:*

- (1) *The isoperimetric profile I is non decreasing, and $\text{Ch}(X) = \lim_{t \rightarrow \infty} I'_+(t) = \lim_{t \rightarrow \infty} I'_-(t)$, where I'_+ , I'_- denote the right and left derivatives of I .*
- (2) *If X is diffeomorphic to \mathbb{R}^3 , then I is strictly increasing and $\text{Ch}(X) < I'_+(t)$, for all $t > 0$.*

Proof. Let X be a non-compact, simply connected homogeneous three-manifold. The isoperimetric profile I of X is non-decreasing (and strictly increasing if X is diffeomorphic to \mathbb{R}^3) by Lemma 2.2. As $I'_+(t)$, $I'_-(t)$ are the mean curvatures of isoperimetric domains for every $t > 0$ by Lemma 2.2, then the remaining statements in Corollary 6.4 follow from Theorem 1.4. \square

Proof of Theorem 1.5. Let X be a homogeneous three-manifold diffeomorphic to \mathbb{R}^3 . If X is isometric to $\widetilde{\text{SL}}(2, \mathbb{R})$ with a left invariant metric, then Proposition 5.6 implies the desired properties. Otherwise, X is isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, and then $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ satisfies all the properties in the statement of the theorem (see the explanation before Theorem 6.3). Now the proof is complete. \square

7 Appendix: Constant mean curvature hypersurfaces obtained by deforming the ambient metric

Let $\Sigma \subset W$ be a compact, two-sided, smooth embedded hypersurface in an n -dimensional ambient manifold. Suppose that for a given Riemannian metric g_0 on W , the following properties hold:

- The mean curvature function of (Σ, g_0) (with the induced metric) is a constant $H_0 \in \mathbb{R}$. In particular, we have chosen an orientation on Σ when $H \neq 0$.
- The Jacobi operator $L: C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$ of (Σ, g_0) has one-dimensional kernel, generated by a Jacobi function $\varphi \in C^\infty(\Sigma)$ with $\int_\Sigma \varphi dA_{g_0} \neq 0$.

Let \mathcal{G} be a neighborhood of g_0 in some collection of metrics, so that \mathcal{G} can be considered to be an open set of a Banach manifold.

Remark 7.1. If $W = \widetilde{\text{SL}}(2, \mathbb{R})/\Delta$ as in Lemma 5.3 where W is equipped with the quotient metric $g_0 = g_W$, then \mathcal{G} could be taken to be a small neighborhood of g_W in the space of locally homogeneous metrics on W that descend from left invariant metrics on $\widetilde{\text{SL}}(2, \mathbb{R})$, which according to Proposition 4.2, are parameterized by the open set $\mathcal{M} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \mid \lambda_i > 0\}$. An application of the deformation results of this section appears in the proof of Lemma 5.18 above, where the compact hypersurface Σ described in the previous paragraph is the torus Σ/Δ in (W, g_W) given in item (D) of Proposition 5.3 and \mathcal{G} is considered to be a small ball in the related set of quotient metrics of W centered at g_W .

Fix $\alpha > 0$. In the sequel, we will consider small open neighborhoods of g_0 in \mathcal{G} and of the function zero in $C^{2,\alpha}(\Sigma)$. We will use the notation \mathcal{G}_ε , $C^{2,\alpha}(\Sigma)_\varepsilon$ for these neighborhoods, which will be often changed by smaller ones while keeping the subindex ε .

As Σ is compact, there exists $\varepsilon > 0$ small enough so that given $g \in \mathcal{G}_\varepsilon$ and $u \in C^{2,\alpha}(\Sigma)_\varepsilon$, the g -normal graph of u over Σ defines an embedded $C^{2,\alpha}$ hypersurface $\Sigma_{g,u} \subset W$ which is diffeomorphic to Σ . This means that the map

$$\phi_{g,u}: \Sigma \rightarrow \Sigma_{g,u}, \quad \phi_{g,u}(p) = \exp_p^g(u(p)N_\Sigma^g(p))$$

is a diffeomorphism, where \exp^g is the exponential map on (W, g) and N_Σ^g is the unit normal vector field to $\Sigma \subset (W, g)$ for which the orientation on $\Sigma_{g,u}$ coincides after pull-back through $\phi_{g,u}$ with the original orientation on Σ . We will denote by $H(g, u)$ the mean curvature of $\Sigma_{g,u}$ with respect to N_Σ^g .

Consider the real analytic map

$$\widehat{H}: \mathcal{G}_\varepsilon \times \mathbb{R} \times C^{2,\alpha}(\Sigma)_\varepsilon \rightarrow C^\alpha(\Sigma), \quad \widehat{H}(g, c, u) = c - H(g, u).$$

Thus, $\widehat{H}(g_0, H_0, 0) = 0$. Our goal is to apply to \widehat{H} the Implicit Function Theorem around $(g_0, H_0, 0)$. Note that the zeros of \widehat{H} can be identified with the set of hypersurfaces $\Sigma' \subset W$ sufficiently $C^{2,\alpha}$ -close to Σ , that have constant mean curvature c in nearby ambient spaces (W, g) to (W, g_0) .

Lemma 7.2. *In the above situation, the differential*

$$(D\widehat{H})_{(g_0, H_0, 0)}: T_{g_0}\mathcal{G} \times \mathbb{R} \times C^{2,\alpha}(\Sigma) \rightarrow C^\alpha(\Sigma)$$

is surjective.

Proof. We will use the standard notation $\frac{\partial \hat{H}}{\partial g} = D_1 \hat{H}$, $\frac{\partial \hat{H}}{\partial c} = D_2 \hat{H}$, $\frac{\partial \hat{H}}{\partial u} = D_3 \hat{H}$ for partial derivatives. Given $(\dot{g}, a, v) \in T_{g_0} \mathcal{G} \times \mathbb{R} \times C^{2,\alpha}(\Sigma)$, we have

$$(D\hat{H})_{(g_0, H_0, 0)}(\dot{g}, a, v) = \left(\frac{\partial \hat{H}}{\partial g} \right)_{(g_0, H_0, 0)}(\dot{g}) + a - Lv. \quad (7.1)$$

Given $w \in C^\alpha(\Sigma)$, define $a \in \mathbb{R}$ by the formula

$$a = \frac{\int_\Sigma w \varphi dA_{g_0}}{\int_\Sigma \varphi dA_{g_0}}.$$

Thus, $a - w$ is orthogonal to φ in $L^2(\Sigma, g_0)$. Since the Jacobi operator $L: C^{2,\alpha}(\Sigma) \rightarrow C^\alpha(\Sigma)$ is self-adjoint with respect to the Hilbert space $L^2(\Sigma, g_0)$ and φ generates the kernel of L , then we conclude that there exists $v \in C^{2,\alpha}(\Sigma)$ such that $Lv = a - w$. Finally, (7.1) gives

$$(D\hat{H})_{(g_0, H_0, 0)}(0, a, v) = a - Lv = w,$$

which proves the lemma. \square

By Lemma 7.2 and the Implicit Function Theorem, there exists $\varepsilon > 0$ small enough so that the set

$$\mathcal{M} = \hat{H}^{-1}(0) = \{(g, c, u) \in \mathcal{G}_\varepsilon \times (H_0 - \varepsilon, H_0 + \varepsilon) \times C^{2,\alpha}(\Sigma)_\varepsilon \mid H(g, u) = c\}$$

is a real analytic manifold passing through $(g_0, H_0, 0)$. Furthermore, the tangent space to \mathcal{M} at $(g_0, H_0, 0)$ is

$$\begin{aligned} T_{(g_0, H_0, 0)} \mathcal{M} &= \text{kernel}(D\hat{H})_{(g_0, H_0, 0)} \\ &\stackrel{(7.1)}{=} \left\{ (\dot{g}, a, v) \in T_{g_0} \mathcal{G} \times \mathbb{R} \times C^{2,\alpha}(\Sigma) \mid \left(\frac{\partial \hat{H}}{\partial g} \right)_{(g_0, H_0, 0)}(\dot{g}) + a = Lv \right\}. \end{aligned}$$

Consider the natural projection

$$\Pi: \mathcal{G} \times \mathbb{R} \times C^{2,\alpha}(\Sigma) \rightarrow \mathcal{G}, \quad \Pi(g, c, u) = g.$$

In the next proposition we prove that every metric $g \in \mathcal{G}$ sufficiently close to g_0 admits a real analytic curve of hypersurfaces $t \in (-\delta, \delta) \mapsto \Sigma_{g, u(g, t)}$ with constant mean curvature $c(g, t)$, which form a deformation of the original hypersurface Σ_0 .

Remark 7.3. In our special setting $W = \widetilde{\text{SL}}(2, \mathbb{R})/\Delta$, this curve of deformed hypersurfaces with constant mean curvature $c(g, t)$ turns out to be a family of leaves of a $c(g)$ -foliation of (W, g) by the statement of Lemma 5.7. In particular in this case, $c(g, t) = c(g)$ does not depend on t . This fact that $c(g, t)$ depends solely on g and not on t in this particular application of Proposition 7.4 below reflects the fact that $(\widetilde{\text{SL}}(2, \mathbb{R}), g)$ is a homogeneous space, in contrast with the framework of this appendix where no homogeneity is assumed.

Proposition 7.4 (Openness). *In the above situation, the differential*

$$[D(\Pi|_{\mathcal{M}})]_{(g_0, H_0, 0)} : T_{(g_0, H_0, 0)}\mathcal{M} \rightarrow T_{g_0}\mathcal{G}$$

is surjective and its kernel is $\{0\} \times \{0\} \times \text{Span}(\varphi)$. In particular:

1. $\dim \mathcal{M} = \dim \mathcal{G} + 1$.
2. *There exist $\varepsilon, \delta > 0$ and a real analytic map*

$$(g, t) \in \mathcal{G}_\varepsilon \times (-\delta, \delta) \mapsto (c(g, t), u(g, t)) \in (H_0 - \varepsilon, H_0 + \varepsilon) \times C^{2,\alpha}(\Sigma)_\varepsilon$$

with $(c(g_0, 0), u(g_0, 0)) = (H_0, 0)$, such that $\{(g, c(g, t), u(g, t)) \mid g \in \mathcal{G}_\varepsilon, |t| < \delta\}$ is an open neighborhood of $(g_0, H_0, 0)$ in \mathcal{M} . In particular, for each $g \in \mathcal{G}_\varepsilon$ fixed, $t \in (-\delta, \delta) \mapsto \Sigma_{g, u(g, t)}$ is a 1-parameter family of compact hypersurfaces of constant mean curvature $c(g, t)$ in (W, g) .

Proof. We first prove that $[D(\Pi|_{\mathcal{M}})]_{(g_0, H_0, 0)}$ is surjective. To see this, take an element $\dot{g} \in T_{g_0}\mathcal{G}$. Define

$$\widehat{w} = \widehat{w}(\dot{g}) = \left(\frac{\partial \widehat{H}}{\partial g} \right)_{(g_0, H_0, 0)}(\dot{g}) \in C^\alpha(\Sigma), \quad (7.2)$$

and

$$a = a(\dot{g}) = - \frac{\int_\Sigma \widehat{w} \varphi dA_{g_0}}{\int_\Sigma \varphi dA_{g_0}} \in \mathbb{R}. \quad (7.3)$$

Then, (7.3) gives that $\widehat{w} + a$ is orthogonal to φ in $L^2(\Sigma, g_0)$. Reasoning as in the proof of Lemma 7.2, we deduce that there exists $v \in C^{2,\alpha}(\Sigma)$ such that $Lv = \widehat{w} + a$. Finally,

$$\left(\frac{\partial \widehat{H}}{\partial g} \right)_{(g_0, H_0, 0)}(\dot{g}) + a = \widehat{w} + a = Lv,$$

thus $(\dot{g}, a, v) \in T_{(g_0, H_0, 0)}\mathcal{M}$. Since clearly $(D\Pi)_{(g_0, H_0, 0)}(\dot{g}, a, v) = \dot{g}$, then we deduce that $[D(\Pi|_{\mathcal{M}})]_{(g_0, H_0, 0)}$ is surjective, as desired.

Now suppose that $(\dot{g}, a, v) \in \text{kernel}[D(\Pi|_{\mathcal{M}})]_{(g_0, H_0, 0)}$. Thus, $\dot{g} = 0$ and $a = Lv$. In particular, a lies in the image of L . As L is self-adjoint with respect to g_0 , then a is L^2 -orthogonal to the kernel of L , which is spanned by φ . Thus, $0 = \int_\Sigma a \varphi dA_{g_0} = a \int_\Sigma \varphi dA_{g_0}$. As $\int_\Sigma \varphi dA_{g_0} \neq 0$, then $a = 0$. Therefore, $\text{kernel}[D(\Pi|_{\mathcal{M}})]_{(g_0, H_0, 0)} \subset \{0\} \times \{0\} \times \text{Span}\{\varphi\}$. The reverse containment is a direct consequence of the above description of $T_{(g_0, H_0, 0)}\mathcal{M}$, which finishes the proof of the first sentence in the statement of the proposition. Item (1) is now obvious, and (2) is a direct consequence of the Implicit Function Theorem applied to $\Pi|_{\mathcal{M}}$. \square

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References

- [1] C. Bavard and P. Pansu. Sur le volume minimal de \mathbb{R}^2 . *Ann. scient. Éc. Norm. Sup.*, 19(4):479–490, 1986. MR0875084 (88b:53048), Zbl 0611.53038.
- [2] S. Gallot. Inégalités isopérimétriques et analytiques sur les variétés Riemanniennes. *Asterisque*, 163–164:31–91, 1988. MR0999971, Zbl 0674.53001.
- [3] F. F. Hoke III. Lie groups that are closed at infinity. *Trans. Amer. Math. Soc.*, 313(2):721–735, 1989. MR0935533, Zbl 0678.58031.
- [4] W. H. Meeks III, P. Mira, J. Pérez, and A. Ros. Constant mean curvature spheres in homogeneous 3-manifolds. Work in progress.
- [5] W. H. Meeks III and J. Pérez. Constant mean curvature surfaces in metric Lie groups. In *Geometric Analysis*, volume 570, pages 25–110. Contemporary Mathematics, edited by J. Galvez, J. Pérez, 2012. Preprint available at <http://www.ugr.es/local/jperez/papers/papers.htm>.
- [6] W. H. Meeks III and G. Tinaglia. Existence of regular neighborhoods for H -surfaces. Preprint available at arXiv 1002.4972, to appear in *Illinois J. of Math.*
- [7] W. H. Meeks III and G. Tinaglia. The CMC dynamics theorem in \mathbb{R}^3 . *J. of Differential Geom.*, 85:141–173, 2010.
- [8] R. Pedrosa. The isoperimetric problem in spherical cylinders. *Annals of Global Analysis and Geometry*, 26(4):333–354, 2004. MR2103404, Zbl 1082.53066.
- [9] N. Peyerimhoff and E. Samiou. The Cheeger constant of simply connected, solvable Lie groups. *Proc. Amer. Math. Soc.*, 132(5):1525–1529, 2004. MR2053361, Zbl 1045.53033.
- [10] A. Ros. The isoperimetric problem. In *Global theory of minimal surfaces*, pages 175–209. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167260, Zbl 1125.49034.
- [11] H. Rosenberg, R. Souam, and E. Toubiana. General curvature estimates for stable H -surfaces in 3-manifolds and applications. *J. Differential Geom.*, 84(3):623–648, 2010. MR2669367, Zbl 1198.53062.